The geodesic hypothesis

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The normal way in which textbooks introduce Einstein's Theory of General Relativity is via the *geodesic hypothesis*. This is the hypothesis that small 'freely-falling' bodies move along geodesic trajectories (i.e. the curved space equivalent of straight lines), see e.g. Hughston and Tod (1990). This hypothesis is usually itself derived from the *equivalence principle*, i.e. the assumption that it is impossible to differentiate in physical experiments confined to infinitesimally small regions of space time between two different inertial frames of reference. Indeed, Einstein himself articulated the principle of local equivalence in 1911:

> *"We arrive at a very satisfactory interpretation of this law of experience, if we assume that the systems K and K' are physically exactly equivalent, that is, if we assume that we may just as well regard the system K as being in a space free from gravitational fields, if we then regard K as uniformly accelerated. This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, just as the usual theory of relativity forbids us to talk of the absolute velocity of a system; and it makes the equal falling of all bodies in a gravitational field seem a matter of course."*

A cornerstone of the argument that many people use to postulate the existence of 'dark matter' is that the trajectories that objects (including light rays) take in certain large astronomical systems appear to deviate from the geodesic hypothesis if one merely takes account of the amount of visible matter that appears to be within them. Arguably there are only two broad ways in which we can explain deviation from Newtonian trajectories of slowly moving objects in weak gravitational fields derived from a certain amount of visible matter. These are if there is some additional unaccounted for 'dark' matter (which alters the magnitude of the gravitational field in the observed manner) or if there is a breakdown in the geodesic hypothesis.

The argument may be developed as follows. Consider why it is that we equate the constant of proportionality, G, in Einstein's field equations with the Newtonian gravitational constant. Typically the linkage is derived along the following lines, see e.g. Hughston and Tod (1980):

(a) In weak gravitational fields we assume locally that we can find a coordinate system that is in a suitable sense approximately Cartesian, so that the metric tensor g_{ab} in this coordinate system is equal to the Minkowski (flat-space) metric plus a smaller term, say, $g_{ab} = \eta_{ab} + \varepsilon h_{ab}$, where η_{ab} is the flat-space metric (so 'weak' here means ε is small). We also assume that with respect to this coordinate system we have for all functions, f, of interest for $i = 1,2,3$ (here $i = 0$ corresponds to the time dimension):

$$\frac{\partial f}{\partial t} = O(\varepsilon) \times \frac{\partial f}{\partial x^i}$$

(b) The 'slow-motion approximation' then leads us to conclude that for a slowly moving particle the geodesic equations reduce to the following, where Γ are the Christoffel symbols:

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{00} \frac{dx^0}{ds} \frac{dx^0}{ds} = O(\varepsilon^2)$$

(c) But (a) implies that $\Gamma^i_{00} = \varepsilon h_{0,i}/2 + O(\varepsilon)$, using Hughston and Tod's notation, so the geodesic equation reduces to:

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}\varepsilon h_{00}$$

This equation equates to Newtonian gravitational theory only if $g_{00} = 1 + \varepsilon h_{00} = 1 + 2\Phi + O(\varepsilon^2)$ where Φ is the Newtonian gravitational potential.

Hughston and Tod (1980) note that "*any* theory of gravitation that incorporates the geodesic hypothesis [i.e. that particles travel along geodesics] must make [this identification], so that this is not a *unique* characteristic of general relativity". Indeed, any theory of gravitation (including General Relativity) that adheres to the geodesic hypothesis will be approximately Newtonian for objects satisfying the slow motion approximation and moving in weak gravitational fields.

References

Hughston, L.P. and Tod, K.P. (1990). *An Introduction to General Relativity*. Cambridge University Press