The Lanczos Approximation

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The Lanczos approximation is a method for computing the gamma function numerically, originally derived by Cornelius Lanczos in 1964 and involving the following formula:

$$\Gamma(z+1) \equiv \int_0^\infty t^z e^{-t} dz = \sqrt{2\pi}\left(z+g+\frac{1}{2}\right)^{z+\frac{1}{2}} e^{-\left(z+g+\frac{1}{2}\right)} L_g(z)$$

Here g is an arbitrary constant subject to the restriction that $Re(z+g+1/2)>0$ and $L_g(z)$ and $p_i(g)$ are as follows, where $C(i,j)$ is the (i,j)'th element of the Chebyshev polynomial coefficient matrix:

$$L_g(z) = \frac{1}{2}p_0(g) + p_1(g)\frac{z}{z+1} + p_2(g)\frac{z(z-1)}{(z+1)(z+2)} + \cdots$$

$$p_k(g) = \sum_{j=0}^{k} C(2k+1,2j+1)\frac{\sqrt{2}}{\pi}\left(j-\tfrac{1}{2}\right)!\left(j+g+\tfrac{1}{2}\right)^{-\left(j+\frac{1}{2}\right)} e^{j+g+\frac{1}{2}}$$

The series L_g converges. By choosing an appropriate g, typically a small positive number, only a few terms are needed to calculate the gamma function to a high degree of accuracy. The series approximation can then be recast into the following form, with the c_k calculated in advance:

$$L_g(z) = c_0 + \sum_{k=0}^{N} \frac{c_k}{z+k}$$

According to Wikipedia (2015), Lanczos derived the formula by deriving the following integral representation for the gamma function and then deriving a series expansion for the integral within this representation:

$$\Gamma(z+1) = (z+g+1)^{z+1} e^{-(z+g+1)} \int_0^e \left(v(1-\log v)\right)^{z-\frac{1}{2}} v^g dv$$