Interpolation and Extrapolation: Introduction

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Suppose we know the value that a function $f(x)$ takes at a set of points $\{x_0, x_1, \ldots, x_{N-1}\}$ say, where we have ordered the x_i so that $x_0 < x_1 < \cdots < x_{N-1}$. However we do not have an analytic expression for $f(x)$ that allows us to calculate it at an arbitrary point. Often the x_i's are equally spaced, but not necessarily. How might we best estimate $f(x)$ for arbitrary x by in some sense drawing a smooth curve through and potentially beyond the x_i? If $x_0 < x < x_{N-1}$, i.e. within the range spanned by the x_i then this problem is called *interpolation*, otherwise it is called *extrapolation*.

To do this we need to model $f(x)$, between or beyond the known points, by some plausible functional form. It is relatively easy to find pathological functions that invalidate any given interpolation scheme, so there is no single 'right' answer to this problem. Approaches that are often used in practice involve modelling $f(x)$ using polynomials or rational functions (i.e. quotients of polynomials). Trigonometric functions, i.e. sines and cosines, can also be used, giving rise to so-called Fourier methods.

The approach used can be 'global', in the sense that we fit to all points simultaneously (giving each in some sense 'equal' weight in the computation). More commonly, the approach adopted is 'local', in the sense that we give greater weight in the curve fitting to points 'close' to the value of x in which we are interested.

A simple example of a 'local' interpolation approach would be *linear interpolation*, implemented in the Nematrian website via the function MnLinearInterpolation. This involves finding the two points straddling the value of x in which we are interested, say x_i and x_{i+1} and identifying as the interpolated value the y for which (x, y) lies along the straight line joining $\big(x_i, f(x_i)\big)$ and $\big(x_{i+1}, f(x_{i+1})\big)$. In such an approach we give no weight at all to any of the $f(x_i)$ other than the two corresponding to the two x_i straddling the value of x in question. More generally, we might use an approach that incorporates only the M nearest neighbours to the value of x in question. The number M of points used, minus 1, is called the order of the interpolation scheme, so linear interpolation is a first order scheme. Increasing the order does not necessarily increase accuracy, especially with polynomial interpolation. Higher-order polynomials can often oscillate wildly between the tabulated points.

Simple nearest-neighbour arrangements in which we give no weight to any points beyond the M nearest ones have a potentially significant weakness. The first and higher order derivatives of the resulting answers are discontinuous wherever the set of points deemed 'local' changes. To overcome this problem, practitioners often use spline functions, in which non-locality is introduced in a way that guarantees global smoothness in the interpolated function up to some given order of derivative.

Interpolation can be done in more than one dimension. Often this is accomplished by a sequence of one-dimensional interpolations, but there are other techniques applicable to scattered data, see e.g. Press *et al.* (2007).