Estimating operational risk capital requirements assuming data follows a bi-exponential distribution

[Nematrian website page: ERMMTOperationalRiskCapitalBiExponentialMoM, © Nematrian 2015]

Suppose a risk manager believes that an appropriate model for a particular type of operational risk exposure involves the loss, $X \geq 0$, coming 50% of the time from an exponential distribution with parameter $\lambda_1 > 0$ and 50% of the time come from an exponential distribution with parameter $\lambda_2 > 0$. The exponential distribution $Exp(\lambda)$ has a probability density function, $pdf(x)$, mean, $E(x)$, and variance $E\left(\left(x - E(x)\right)^2\right)$ as follows:

$$pdf(x) = \lambda \exp(-\lambda x) \qquad 0 \leq x \leq \infty$$
$$E(x) = \frac{1}{\lambda}$$
$$E\left(\left(x - E(x)\right)^2\right) = \frac{1}{\lambda^2}$$

Suppose we want method of moments (MoM) estimators for λ_1 and λ_2 and we have loss data X_i for $i = 1, \dots, n$ where the number of losses, n, is sufficiently large to be able to ignore small sample corrections. Then the MoM estimators can be derived as follows, where the (sample) moments used in the estimation are $A = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $B = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

The pdfs of the individual parts, $f_1(x)$ $f_2(x)$, and of the overall distribution, $f(x)$, are:

$$f_1(x) = \lambda_1 \exp(-\lambda_1 x) \qquad f_2(x) = \lambda_2 \exp(-\lambda_2 x)$$
$$f(x) = \frac{1}{2}\left(f_1(x) + f_2(x)\right) = \frac{\lambda_1}{2}\exp(-\lambda_1 x) + \frac{\lambda_2}{2}\exp(-\lambda_2 x)$$

The mean of $f(x)$ is:

$$\mu = \int_0^\infty x \frac{1}{2}\left(f_1(x) + f_2(x)\right) dx = \frac{1}{2}\left(E_1(x) + E_2(x)\right) = \frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)$$

where $E_1(x) = \int_0^\infty x f_1(x) dx$ and $E_2(x) = \int_0^\infty x f_2(x) dx$

By integrating by parts or by noting that $\int_0^\infty x^2 f_j(x) dx = \int_0^\infty \left(x - E_j(x)\right)^2 f_j(x) dx + \left(E_j(x)\right)^2 = k_j^2 + k_j^2 = 2k_j^2$ for $j = 1,2$ where $k_1 \equiv 1/\lambda_1$ and $k_2 \equiv 1/\lambda_2$, we note that

$$\int_0^\infty x^2 \frac{1}{2}\left(f_1(x) + f_2(x)\right) dx = \frac{1}{2}(2k_1^2 + 2k_2^2) = k_1^2 + k_2^2$$

So method of moments estimators involve (if these simultaneous equations have a (real) solution):

$$\frac{1}{2}(k_1 + k_2) = A$$
$$k_1^2 + k_2^2 = B$$
$$\Rightarrow \quad k_1^2 + (2A - k_1)^2 = B$$
$$\Rightarrow \quad 2k_1^2 - 4Ak_1 + (4A^2 - B) = 0$$

This is a quadratic equation which has the following solutions:

$$k_1 = \frac{4A \pm \sqrt{16A^2 - 8(4A^2 - B)}}{4} = A \pm \sqrt{\frac{B}{2} - A^2}$$

The two roots correspond to k_1 and k_2 (it is not possible to differentiate between them given merely the data being provided). Hence the values of λ_1 and λ_2 are:

$$\lambda_j = \frac{1}{A \pm \sqrt{B/2 - A^2}}$$

In practice it is more likely that the probabilities of drawing from the underlying exponentials are unknown. This adds an extra degree of freedom which would introduce the need to include a further (higher) moment into the parameter estimation process.