Showing that the Mean Excess Function of a Generalised Pareto Distribution is linear in the exceedance threshold (for a specific range of values of the distribution's shape parameter)

If a random variable, X, is distributed according to a generalised Pareto distribution, $X \sim GPD(\xi,\mu,\sigma)$, then it has the following probability density function (for $\sigma > 0$):

$$f(x) = \begin{cases} \frac{1}{\sigma}\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1-1/\xi} & \xi \neq 0 \\ \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right) & \xi = 0 \end{cases}$$

If $\xi \geq 0$ then its domain is $\mu \leq x < +\infty$.

The mean excess function of a probability distribution is defined as:

$$mean\ excess\ function = e(u) = E(X-u|X>u)$$

If $0 < \xi < 1$ then then mean excess function for this distribution is as follows (for $u \geq \mu$):

$$e(u) = \frac{\int_u^\infty (x-u)\frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1-\frac{1}{\xi}}dx}{\int_u^\infty \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1-\frac{1}{\xi}}dx}$$

Let $y = 1+\xi(x-\mu)/\sigma$ so $dy/\xi = dx/\sigma$ and $x = \mu - \sigma/\xi + \sigma y/\xi$. Let $w = 1+\xi(u-\mu)/\sigma$. Then:

$$e(u) = \frac{\int_w^\infty \left(\frac{\sigma y}{\xi}+\mu-\frac{\sigma}{\xi}-u\right)\frac{1}{\xi}y^{-1-\frac{1}{\xi}}dy}{\int_w^\infty \frac{1}{\xi}y^{-1-\frac{1}{\xi}}dy} = \frac{\sigma}{\xi}\frac{\int_w^\infty y^{-\frac{1}{\xi}}dy}{\int_w^\infty y^{-1-\frac{1}{\xi}}dy}+\mu-u-\frac{\sigma}{\xi}$$

$$\Rightarrow e(u) = \frac{\sigma}{\xi}\frac{\left[\frac{y^{1-\frac{1}{\xi}}}{1-\frac{1}{\xi}}\right]_w^\infty}{\left[\frac{y^{-\frac{1}{\xi}}}{-\frac{1}{\xi}}\right]_w^\infty}+\mu-\frac{\sigma}{\xi}-u = \frac{\sigma}{\xi}\frac{-\frac{1}{\xi}}{1-\frac{1}{\xi}}\frac{\left(0-w^{1-\frac{1}{\xi}}\right)}{\left(0-w^{-\frac{1}{\xi}}\right)}+\mu-u-\frac{\sigma}{\xi}$$

$$\Rightarrow e(u) = \frac{\sigma w}{\xi(1-\xi)}+\mu-\frac{\sigma}{\xi}-u = \frac{\sigma}{\xi(1-\xi)}\left(1+\frac{\xi(u-\mu)}{\sigma}\right)+\mu-u-\frac{\sigma}{\xi}$$

$$\Rightarrow e(u) = \frac{\sigma}{1-\xi}+\frac{\xi}{1-\xi}(u-\mu)$$

This is linear in u as desired. A consequence is that we can test visually whether a data set appears to be coming from a GPD by plotting the empirical mean excess function and seeing if it appears to be linear (and we can also estimate $1/(1-\xi)$ from its slope if it is linear).