Identifying a formula for the (lower) conditional tail expectation (CTE) of a normal distribution that does not explicitly include integral signs but instead refers to the unit normal density function and the unit normal cumulative distribution function

[Nematrian website page: ERMMTFormulaForCTEWithoutIntegralSigns, © Nematrian 2015]

The (lower) conditional tail expectation (CTE) of a random variable for a given cut-off, q, is defined as the expected value of the random given that it is below the relevant cut-off. It is thus very closely aligned with the Tail Value-at-Risk (TVaR) of the distribution, since the TVaR is the CTE for a specific cut-off. For a normal distribution the CTE is thus:

$$CTE(q) = E(X|X \leq q) = \frac{\int_{-\infty}^{q} xp(x)dx}{\int_{-\infty}^{q} p(x)dx} = \frac{\int_{-\infty}^{q} \frac{x}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx}{\int_{-\infty}^{q} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx}$$

The unit normal distribution function, $\phi(x)$, and the unit Normal cumulative distribution function, $N(x)$, are defined as follows:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$
$$N(x) = \int_{-\infty}^{x} \phi(y)dy$$

Substituting $z = (x-\mu)/\sigma$ in the above formula for the CTE we obtain, where $b(q) = (q-\mu)/\sigma$):

$$CTE(q) = \frac{\int_{-\infty}^{b(q)} (\mu + \sigma z)\phi(z)dz}{N(b)}$$

We can re-express this in a form that relies only on $N(x)$ and $\phi(x)$ by noting the following:

$$\frac{d}{dz}\phi(z) = \frac{d}{dz}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) = -\frac{z}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$
$$\Rightarrow \phi\big(b(q)\big) = C - \int_{-\infty}^{b(q)} z\phi(z)dz$$

where C is constant and furthermore as $\phi(-\infty) = 0$ and $z \exp\left(-\frac{z^2}{2}\right)$ decays sufficiently fast to zero as $z \to -\infty$ we have $C = 0$. Hence:

$$\int_{-\infty}^{b(q)} (\mu + \sigma z)\phi(z)dz = \mu N\big(b(q)\big) - \sigma\phi\big(b(q)\big)$$
$$\Rightarrow CTE(q) = \frac{\mu N\big(b(q)\big) - \sigma\phi\big(b(q)\big)}{N\big(b(q)\big)} = \mu - \frac{\sigma\phi\big(b(q)\big)}{N\big(b(q)\big)}$$