Chebyshev Polynomials

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The Chebyshev polynomial of degree n is denoted $T_n(x)$. It is defined as:

$$T_n(x) \equiv \cos(n \cos^{-1} x) \qquad -1 \leq x \leq 1$$

Chebyshev polynomials satisfy the following relationships and recursion formulae:

Recursion formulae:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad n = 2,3, \dots$$

Zeros, exactly n distinct zeros in the interval $[-1,1]$, as follows:

$$T_n(x) = 0 \text{ for } x_k = \cos\left(\frac{(2k+1)\pi}{2n}\right) \quad k = 0,1, \dots n-1$$

Extrema, exactly $n+1$ real extrema in the interval $[-1,1]$, as follows:

$$|T_n(x)| \leq 1 \qquad -1 \leq x \leq 1$$
$$T_n(x_k) = (-1)^k \; for \; x_k = \cos\frac{k\pi}{n} \quad k = 0, \dots n$$

Orthogonality:

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{(1-x^2)^{1/2}} dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi/2 & \text{for } m = n \neq 0 \\ \pi & \text{for } m = n = 0 \end{cases}$$