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ASSET/LIABILITY MODELLING FOR PENSION FUNDS

by

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ABSTRACT

This paper describes some of the problems that arise when carrying out asset/liability studies of UK final salary pension schemes, especially if the investment strategies being tested include derivatives or other sorts of 'dynamic' investment strategies. It considers the choice of the underlying stochastic investment model in some detail and concludes that models incorporating mean reverting characteristics (such as the Wilkie model) are likely to be inappropriate in these circumstances. It presents more robust alternatives.

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1. INTRODUCTION

1.1 Asset/liability modelling has become quite common for UK pension schemes over the last few years. Its use is likely to increase further as a result of the introduction of the *Minimum Funding Requirement* (MFR) contained in the Pensions Act 1995.

1.2 The basic rationale behind such studies is that different pension schemes have different needs (since each scheme has different liabilities and objectives). Hence they potentially need different investment strategies.

For example, the MFR will impose sanctions on pension schemes whose 'funding levels' (calculated according to the rules laid down in the MFR) fall below certain trigger points. In such circumstances, the sponsoring employer is required to pay in extra contributions into the scheme. Trustees (or the sponsoring employer) may want to know whether the current investment strategy is likely to incur an unacceptable risk of falling below these trigger points, and whether there is merit in changing investment strategy. The risks involved will differ by scheme and what is 'unacceptable' to one set of trustees may be more acceptable to others.

Changing investment strategy may also affect the magnitude (and volatility) of more regular contributions paid into the scheme by the sponsoring company, or the pension cost recognised in the company's profit and loss account. Finance directors may have a keen interest in the results of analyses focusing on these sorts of statistics.

1.3 Investment policy is often intimately bound up with *funding policy*, i.e. decisions on how rapidly to fund for the benefits accruing within the scheme. Asset/liability studies may therefore analyse the impact of changing funding policy as well as (or instead of) investment policy.

1.4 To date most studies have concentrated on essentially static investment strategies. Extending the range of strategies under consideration to include dynamic ones has profound implications for asset/liability studies, as we shall see in later sections.

2. THE TYPICAL RESULTS OF AN ASSET/LIABILITY STUDY

2.1 The term *asset/liability modelling* means different things to different people. Banks typically view it as one of the processes they might use to ensure that any mismatches between assets and liabilities are not so large as to expose them to serious risk if there is a sudden sharp market movement in the *near* future.

2.2 For long-term investing institutions such as pension funds and insurance companies, asset/liability modelling more usually refers to the carrying out of projections of asset and liabilities (and of related characteristics, e.g. solvency levels) over periods of at least several years. It is usually associated with carrying out many different projections under different scenarios, particularly in the form of *stochastic* modelling. This may be contrasted with *deterministic* modelling, which involves looking at just one (or possibly a small number) of projections.

2.3 Although pension fund asset/liability studies can test the impact of changing funding policy, it is more common for them to concentrate on reviewing investment policy. Usually the final result of the study (apart from hopefully a better understanding by all concerned of the dynamics of the pension scheme) is a recommended investment strategy for the scheme to follow in the form of either:

(a) a benchmark consisting of specific (fixed) percentages allocated to each asset class, which the investment manager is expected to follow (within suitable ranges); or

(b) extraction of a 'core' portfolio, typically of bonds, with the remaining assets invested as previously, e.g. in a 'balanced' fashion. This can be thought of as a special case of (a) in which there are two asset 'classes', one consisting of the appropriate sort of bonds and one consisting of the remaining balanced portfolio.

2.4 The percentage of the total market value of the portfolio invested in each asset category (or in the core/non-core split) is typically rebalanced back to the (static) benchmark weighting from time to time, e.g. quarterly, annually or in a more ad hoc fashion. Without this, the actual portfolio mix will drift from the benchmark mix that was originally set. Adopting this sort of rebalancing approach introduces some subtle yet potentially very important assumptions regarding the sorts of investment strategies we are prepared to countenance:

(a) It implicitly ignores investment strategies that are *dynamic*, i.e. that vary in time by more than merely market drift; and

(b) It also ignores nearly all investment strategies involving *derivatives*, such as options. Many of the characteristics of dynamic investment strategies can be replicated by derivative strategies and vice versa, see e.g. Kemp (1996).

2.5 It is unlikely that considering merely 'static' sorts of investment strategies is optimal. For example, suppose a scheme wishes to maximise its equity exposure whilst also minimising the risk of its MFR funding level falling below some trigger point. An obvious way of doing this would be to maintain a high equity weighting if the MFR funding level is well above the trigger point, but progressively to reduce the equity exposure as the MFR funding level approaches the trigger point. This sort of strategy, or a similar derivative based strategy, could permit the maintenance of a higher equity exposure than any purely static approach.

3. <u>WHAT STEPS ARE INVOLVED?</u>

3. The main stages in an asset/liability study are usually:

3.1 *The key objectives that investment and funding policy should aim to achieve need to be clarified.*

The key objectives generally relate to the control of features such as:

(a) *future ongoing funding levels.* This is often viewed by actuaries as the best all-round, *long-term* measure of the health of the pension fund.

(b) *future solvency levels* including scheme-specific definitions of solvency or, more probably, MFR funding levels (even though the MFR is not strictly speaking a test of solvency). This is a shorter-term objective than one concentrating on the ongoing funding level. Controlling solvency levels or MFR funding levels may therefore require a different strategy to that which might be ideal for controlling ongoing funding levels. Trustees are likely to be particularly interested in the future position of the scheme vis-a-vis the MFR since they will have to comment on it in the statement of invetment principles they will need to prepare under the Pensions Act 1995.

(c) *future company contribution rates or accounting measures of the cost of providing pension benefits.* Asset/liability studies often include an analysis of SSAP 24 pension costs, because this is the measure of pension cost that impacts on the sponsoring company's bottom line. Subsidiaries of US parents may instead wish to analyse FAS 87 costs.

3.2 *Suitable assumptions to use in the study need to be agreed.*

This may involve input from the actuary, the trustees and the investment managers(s). I have summarised the main sorts of assumptions in Appendix A. The key assumptions are the ones relating to the future behaviour of economic factors underlying the assets and the liabilities as encapsulated in the *stochastic investment model* underlying the projections. The choice of model is considered in depth in Sections 4 to 7.

3.3 *Data needs to be collated to carry out the projections of assets and liabilities.*

Typically data from the latest actuarial valuation is used. The data may be adjusted to reflect known subsequent events, particularly if the valuation is somewhat out of date. Appendix A also contains a summary of the main data items that would need to be collated for a study.

3.4 *The overall nature of the liabilities is considered.*

Some form of broad-brush analysis of the liabilities of the scheme (relative to the average fund) might be carried out, perhaps indicating in broad terms how well funded it is, how mature it is and what its cash flow position is.

For example the scheme illustrated in Figures 1 and 2 is well funded but very mature. The main lesson that might be brought out here is that the maturity of this scheme could be an issue (particularly in the context of the MFR, since its relatively short-term focus tends to make it particularly sensitive to maturity). If the scheme was very well funded and immature, then there would probably be no real constraints on investment policy, and it might be unecessary to carry out any further analyses.

Figure 1. Funding Levels of Schemes of FT-SE 100 Companies

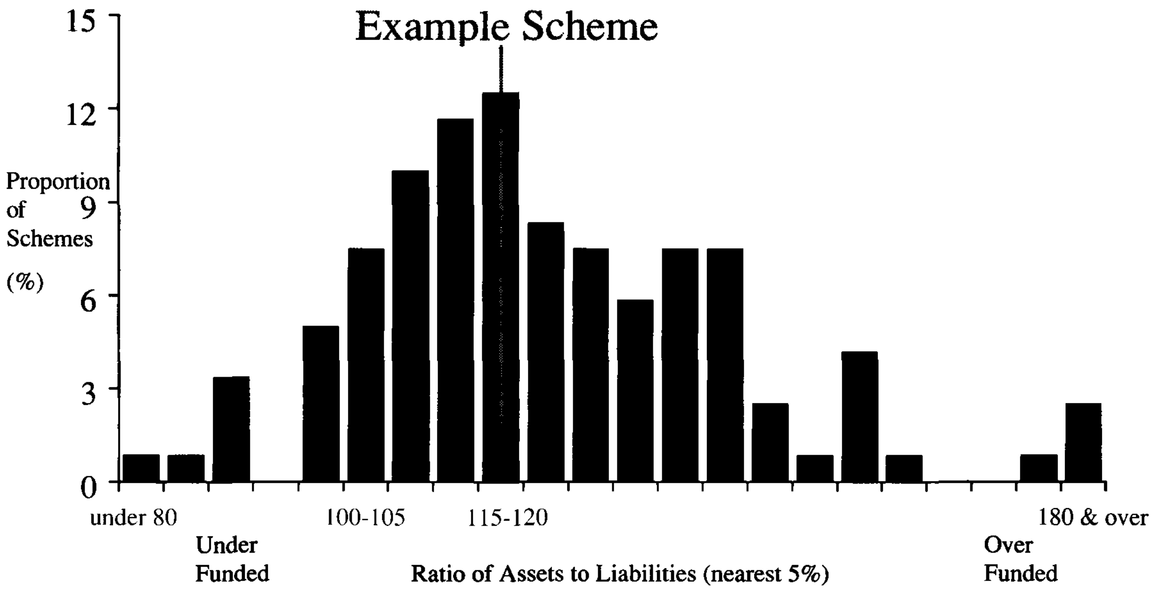
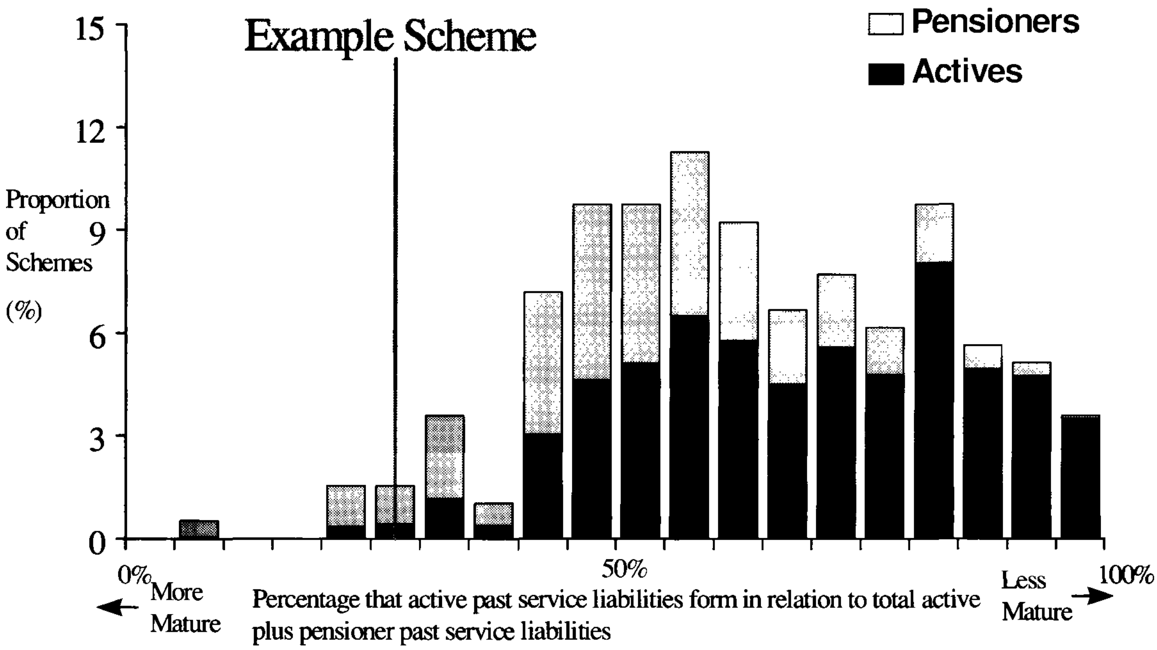


Figure 2. Maturity of Sample Schemes



3.5 *An analysis would be carried out to identify how the scheme might progress in the future if different investment strategies (and possibly funding strategies) were adopted.*

This stage usually involves testing the consequences of many different economic outcomes by Monte Carlo type simulations. It would often concentrate principally on the implications of changing the overall investment strategy, i.e. its equity/bond mix, rather than the fine detail (e.g. the subdivision of the equity or bond parts into their various components).

The precise method of presentation seems to vary, but the usual approach (if considering merely *static* investment strategies) is illustrated in Figures 3 and 4. In these Figures the horizontal axis refers to time whilst the vertical axis refers to, say the MFR funding level at that time. Figure 3 shows what might happen if the current investment strategy continues (in this example, the current investment strategy had 80% in equities), whilst the dotted lines in Figure 4 show what might happen were a less equity-orientated investment strategy adopted.

Figure 3. MFR Funding Level

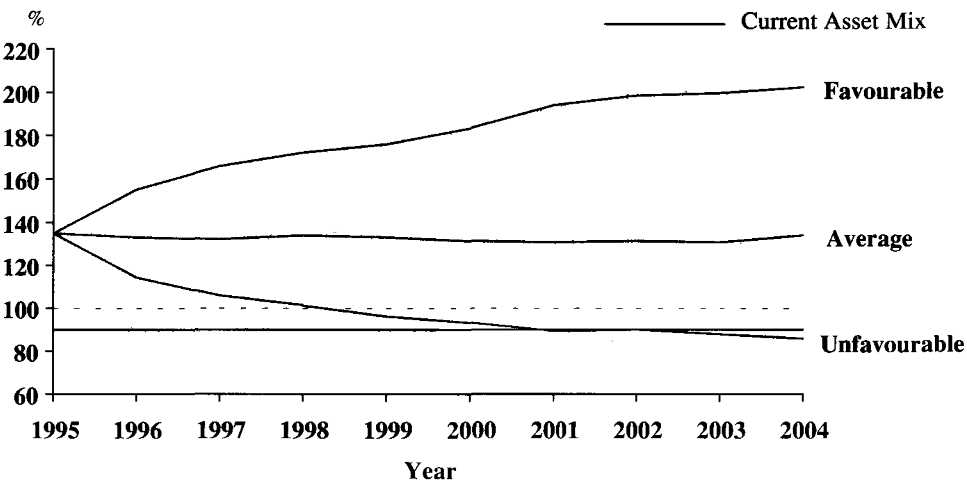
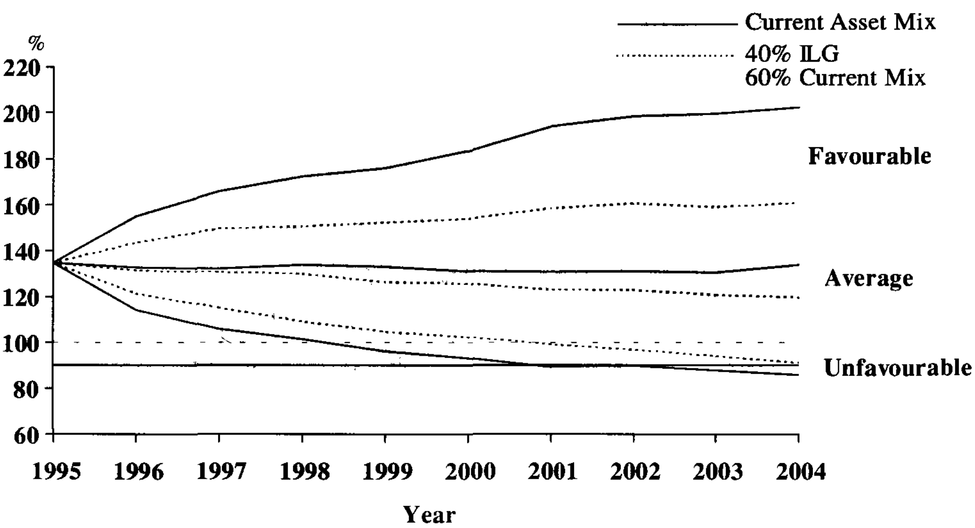


Figure 4. MFR Funding Level (Alternative Strategies)



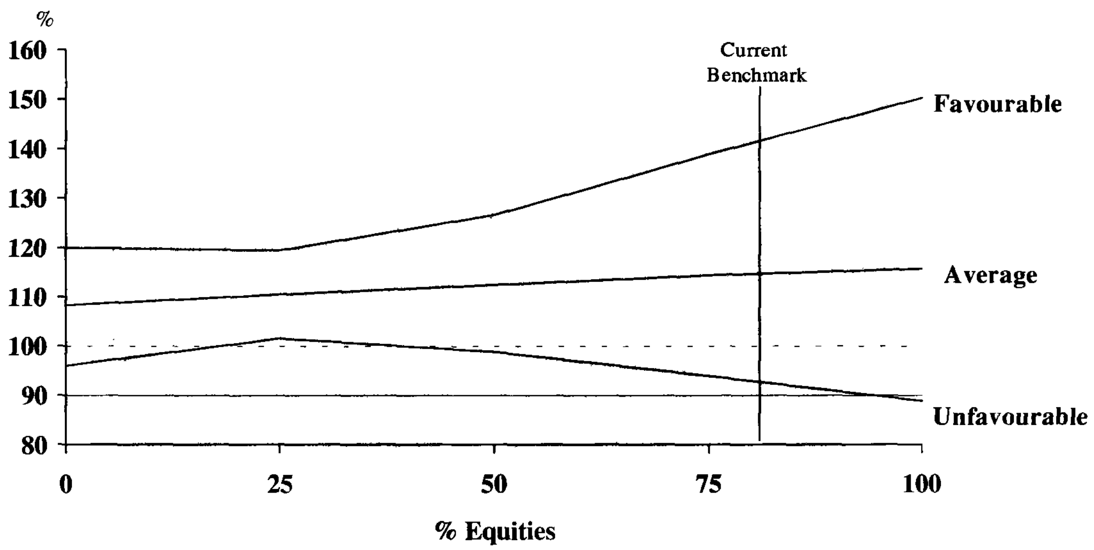
The lines in these graphs might be calculated as follows. A large number of simulations of the future development of the scheme would be carried out. These would be ranked according to the MFR funding level at a given point in time. Some definition of 'favourable', 'average' and 'unfavourable' would be adopted, e.g. the 'favourable' outcome might be the outcome which only 10% of outcomes exceeded, the 'unfavourable' outcome might be the outcome which only 10% of simulations fell below, and the 'average' outcome might be the median. The 'favourable' (top) lines would thus in effect show the outcomes if things go well, whilst the 'unfavourable' (bottom) lines show what might happen if quite adverse circumstances prevail.

In this example, reducing the amount invested in equities from 80% to 50% reduces average outcomes slightly and favourable ones significantly. However, it also has a beneficial impact on unfavourable outcomes (particularly ones falling below the 90% trigger point relevant to the MFR).

It is worth noting that in this sort of presentation the points on, say, the 'unfavourable' line do not necessarily correspond to the same simulated scenario. If the scheme is already on the unfavourable line, say five years into the projection, there is more than a 10% chance that it will be below the unfavourable line in year 6.

A whole range of possible (static) investment strategies could be considered simultaneously, perhaps presented as per Figure 5, which shows how the 'favourable', 'average' and 'unfavourable' outcomes in three year's time might alter if investment strategy is altered now (and kept constant for the next three years). Specific scenarios chosen in conjunction with the trustees might also be tested.

Figure 5. MFR Funding Level in Three Years Time



Significantly different presentational tools would probably need to be developed if the range of investment strategies under consideration is extended to include dynamic strategies. Perhaps this is one reason why few practitioners have tried to extend asset/liability studies in this manner.

3.6 *Different asset mixes would then be analysed in more detail to assess the risks (relative to the liabilities) and the rewards of each alternative under consideration.*

Sometimes, there is no need to analyse further the optimum asset mix for the fund, particularly if the recommended implementation merely involves extraction of a bond component from an existing balanced portfolio.

However, in other instances it may be necessary to consider in more detail the nature of the optimum asset distribution, e.g. the proportion to be invested in UK as opposed to overseas equities, the proportion to invest in property, and the composition of any bond component.

In these instances, this stage would normally involve optimisation techniques designed to identify in more detail the mixtures of asset categories which are most attractive in terms of trading off risk and reward. The most common optimisation approach is to use an approach linked to *efficient frontiers*. These are explained further in Appendix B. It is worth noting that the results of these analyses tend to be quite sensitive to the assumptions adopted, some of which (e.g. those for property) are difficult to determine.

To date these analyses have again tended to concentrate on static asset mixes. However, it is possible to generalise the sorts of techniques referred to cover dynamic strategies, as is described in Appendix B.3

3.7 *The results of the study would then be summarised and presented to the trustees (and perhaps representatives from the sponsoring company).*

Draft versions may be discussed with officers of the scheme before being finalised. For complicated studies it may be necessary to carry out further projections following discussions of preliminary results before finalising the report.

The salient features would be summarised in a presentation, as necessary. The results would typically be discussed with the investment managers at this stage if not before. Their input is likely to be of particular importance if the asset/liability study recommends a change in investment strategy, as they will be able to advise on the best way to make such changes and the best time to carry them out.

3.8 One might argue that a traditional actuarial valuation considers the interaction of assets and liabilities. Why then go to the trouble of carrying out a more detailed asset/liability study?

The answer is that an asset/liability study provides much more information than is available from an actuarial valuation. A valuation provides a single 'answer' at a set point in time (the valuation date). In contrast, an asset/liability study provides three or more extra dimensions by:

(a) providing projections into the future (introducing a *time* dimension);

(b) providing some estimate of the range of likely outcomes (a *probabilistic* dimension); and

(c) indicating the effect of changing investment strategy (an *asset mix* dimension).

4. <u>WHAT ASSUMPTIONS ARE NEEDED?</u>

4.1 One consequence of being able to produce much more information than a traditional actuarial valuation is that an asset/liability study needs more assumptions. We generally need to make assumptions that simplify the calculations involved in the projections and that make it practical to analyse all the key features that the client is interested in. Appendix A sets out the main sorts of simplifying assumptions that are typically made, and how in general terms computer systems are usually constructed to carry out the necessary projections.

4.2 However, these sorts of assumptions are not, in a fundamental sense, particularly important. With sufficiently accurate model construction (and sufficient information on the membership), the impact of any residual simplifying assumptions on scheme benefits can always be made minimal. Given unlimited resources, we could always replicate the scheme benefits exactly, thereby avoiding the need to make assumptions in this area entirely.

4.3 More fundamental are assumptions for which there is no universally accepted 'right' answer. For example, if we wish to project the development of a scheme which has some equity investments, we must make assumptions about the returns that might be achieved on these assets. There will always be some subjectivity or judgement about these assumptions, since it is impossible to predict with certainty how equities will perform in the future.

4.4 The key fundamental assumptions are the ones concerning the future behaviour of economic variables that influence how the scheme's assets and liabilities might change in the future. In totality these assumptions are generally referred to as the *stochastic investment model* underlying the study. For a pension fund asset/liability study the stochastic investment model typically involves assumptions about the future behaviour of the following variables:

(a) price inflation;
(b) salary growth; and
(c) returns on assets (and yields on assets, where relevant).

4.5 It is important in this process to distinguish between projection assumptions and valuation assumptions.

The *projection assumptions* are what we assume will have occurred between the start of the projection and the relevant valuation date. Typically, lots of projections are carried out, chosen in a manner designed to reflect the likely variability of future economic conditions. For example, in some projections RPI inflation will be high throughout the projection period, in others low, and in still others it will start high but then fall or vice-versa. The projection assumptions describe some probability distribution about how we expect the future to evolve.

In contrast, the *valuation basis* is the set of assumptions that we assume will be used by the actuary to carry out an actuarial valuation at the future point in time being considered. In pension fund asset/liability modelling the valuation basis is often assumed to be largely constant, at least for long-term ongoing actuarial valuations. Actuaries do not necessarily alter their assumptions about, say, future long-term inflation merely because in the last two or three years it has been higher or lower than expected. There may also be an element of prudence incorporated within these assumptions, which may not be present in the projection assumptions.

5. AN EXAMPLE OF A STOCHASTIC INVESTMENT MODEL — THE WILKIE MODEL

5.1 Perhaps the best known stochastic investment model in the UK actuarial literature is the Wilkie model, as set out in e.g. Wilkie (1995). Like all others used in practice, it consists of:

(a) a choice of fundamental economic variables, which are then modelled. These include economic variables such as asset returns, price indices, dividend yields etc.

(b) a set of relationships between these fundamental economic variables. These relationships define the value at time t (in the future) of each economic variable A, say $A(t)$, using formulae which depend on:

- random components, because the future is unpredictable. The introduction of random components makes such a model *stochastic*.
- the values of other economic variables, and
- often, previous values of itself, e.g. $A(t\text{-}1)$, $A(t\text{-}2)$,... (but not future values), or of other economic variables. Such processes, which do not look 'forwards', are called *adapted* or *non-anticipating*.

5.2 It is the relationships between the fundamental economic variables that differentiate one economic model from another. Often these relationships are parameterised, meaning that they depend on the values of various additional parameters that the user can vary. The Wilkie model has at least 50 parameters, and so there is in fact no 'one' Wilkie model, rather a family of models depending on the choice of parameters that the user uses.

5.3 The introduction of dependencies on previous values of the same variable is called *autoregression*. The Wilkie model is therefore an *autoregressive* stochastic model. Not all economic models have these autoregressive characteristics (as long as we ignore 'trivial' autoregressive characteristics like the dependency of a rolled up index on previous values of itself).

The Wilkie model (at least its equity component) exhibits a special sort of autoregressive nature. It is *mean reverting*, i.e. if in one year the return on some asset category (or some other economic variable being modelled is less than expected) then this increases the likelihood that it will be greater than expected in the following year.

5.4 In the Wilkie model the economic relationships in Section 5.1(b) are hierarchical, in that 'more fundamental' variables drive ones lower down the hierarchy. Each variable is defined over an annual time period. Ignoring its most recent extensions to overseas equities, the Wilkie model hierarchy is:

(a) RPI inflation, which is assumed to be driven only by random components and previous values of itself;
(b) Wages, which are assumed to be driven by the current (and the immediately preceding) RPI inflation rate and by a random component;
(c) Share dividend yields and, derived from them, the share dividends themselves, from which returns on equities can be derived;
(d) Conventional gilt yields (which Wilkie models on the basis that they relate to consols, i.e. undated government stock), from which returns on gilts may be derived;

(e) Index-linked gilt yields, which with (a) permit derivation of the returns on index-linked gilts;
(f) Property yields and property returns; and
(g) Cash returns.

Items (c),(d), (e) and (f) all depend on (a) (and on random components and autoregressive elements) but not (b). Cash returns, i.e. item (g), are also dependent on gilt yields/returns.

Such a hierarchical approach contains the implicit assumption that the behaviour of factors higher up the hierarchy does not depend on factors lower down. It is doubtful if any such hierarchy is ever fully one way. For example, any link between equities and property will have some two-way nature. Part of the equity market consists of property company shares, but property prices are themselves influenced by company profitability since this affects the ability of companies to pay high rents.

5.5 How does one go about defining the relationships forming such a model? The Wilkie model is based on what are called 'AR1' processes, which are specific forms of autoregressive series showing mean reverting characteristics. This means that, in the absence of any random components, the economic variables tend to revert to some set of long term assumed mean levels. The underlying relationships have been defined partly by application of economic intuition but mainly by fitting relationships to past data using accepted statistical techniques.

5.6 This approach is not without its criticisms from a purely statistical viewpoint. These include:

(a) The model has been criticised as a poor fit with past history, e.g. Huber (1995).

(b) Conversely, in its standard form, it has characteristics drawn from past history which do not obviously fit with intuition. For example, if we project forward the model for the next 20 years lots of times, starting from the model's long-term 'neutral' position as defined in Wilkie (1995), we find that the asset category with the highest expected return is property. Whilst this might please property surveyors, most investment managers and consultants expect equities to outperform property over the long-term (although they may have views on whether either is currently cheap or dear relative to its long-term trend). The reason is that property is a factor of production that forms part of the costs of running a company. In a capitalist economy it is reasonable to suppose that shareholders will get a higher reward for undertaking economic activity than they would do for merely investing in bricks and mortar.

We can, of course, overcome these sorts of problems by altering the parameters used in the model (although this will then worsen its fit to past history). Indeed, this is the option preferred by Professor Wilkie to incorporate such intuitive reasoning.

(c) However, we then face a third problem. The Wilkie model is quite difficult to understand (perhaps though the same could be said of the markets it is trying to model!). Even ignoring extensions to overseas equities, it contains 50 different parameters (or perhaps more, depending on whether one wishes to count aspects of the model which were originally considered variable but which were set to trivial default values). 13 of these describe the 'initial' economic position and 37 define how the position might develop in the future. It is not immediately obvious how important each of the assumptions is or which ones you should alter to accommodate specific economic views such as the one described above.

(d) Related to (b) is the philosophical question of whether the past really is a good guide to the future, or whether it is ever likely to be possible to reconcile past data with intuition. The Wilkie model is based on the principle of time homogeneity, which means that the rule for getting from time t to $t+1$ is the same rule as for getting to time $t+1$ to $t+2$.

We know that this is not actually the case, for example because of the tendency for interest rates to fall and equity markets to be more volatile in the run up to an election. This also means that the historic fitting of the model may be of dubious relevance. It may fail to take into account the one-off nature of many political events significantly influencing investment markets such as the abolition of the gold standard, the second world war and, more recently, the formation and disintegration of the ERM.

Relying exclusively on past history is also more subjective than it looks. Any observed parameters based on past history will be subject to statistical error. There is also the problem of deciding what past history to use. For example, when estimating future inflation rates:

- What period should we use? Inflation rates in the UK averaged over the last few centuries have been close to zero, but significantly positive during most of the 20th century. Do we ignore previous centuries?

- Do we use only UK data or do we assume that experience in other countries might have pointers for the UK. For example, if we think there is even a remote possibility of hyperinflation of the sort experienced in the German Weimar Republic appearing here, then our assumption regarding the long term expected rate of inflation will increase materially!

I personally prefer an approach in which significant weight is placed on intuition (of the consultant, the investment manager(s), the trustees and/or the sponsoring employer).

5.7 Another potential weakness of the Wilkie model is that the random components in the model are assumed to be normally distributed. This is a particular form of probability distribution, with a familiar bell-shaped structure. It is fairly common to use such distributions within econometric modelling. It is also clear that at times extreme movements, such as the October 1987 Crash or the 1973-5 UK equity crash and rebound, do sometimes occur. Wilkie does test to see if the residual random components in his model are normally distributed. In many cases they are not, unless outliers, i.e. extreme values, are removed.

5.8 Nearly all of these criticisms can be applied to *any* stochastic model used for asset/liability modelling. The question of whether to rely on past data or intuition applies to the choice of parameters for any model and no plausible model is likely to fit past data with a very high degree of confidence unless the model is extremely complicated. Economists and others often assume normal probability distributions, even when they do not think that this assumption is rigorously defensible, because it makes the mathematics easier. If the subsequent analysis of the results strips out extreme cases then worrying about the extreme tail of a distribution may not be very fruitful.

5.9 However, there are some rather more fundamental criticisms which *are* specific to the Wilkie model, and become particularly problematic if we wish to test dynamic investment strategies. These relate to the inclusion of mean reverting characteristics into the model.

Mean reverting models often include features that are difficult to reconcile with perceived investment wisdom. For example, in the Wilkie model, UK equity income streams are relatively stable, so that price movements are mainly associated with changes in yields. However, the model also assumes that yields are mean reverting. This means that at times of high yields, the yield is likely to fall causing a price rise, i.e. high yields imply high expected growth rates. Traditional investment wisdom based on discounted cash flow would suggest that the opposite should be the case, since high yields are typically associated with low expected growth rates.

The whole premise of whether markets show mean reverting behaviour can also be questioned. Markets do perhaps seem to overreact to events, and then bounce back, i.e. precisely the behaviour underlying mean reversion. However, they also seem to exhibit 'momentum', albeit over possibly shorter time-scales, i.e. a tendency to continue in an upwards or downwards track. So perhaps there is a danger that we only apply that part of 'perceived investment wisdom' that happens to fit the argument we want to justify! Academic research seems to come to mixed conclusions, with some studies suggesting the existence of mean reversion and others that do not, see e.g. comments in Exley & Mehta (1996).

5.10 Of course, these sorts of criticisms do not of themselves make the model wrong. However, there is a more fundamental objection to the use of these sorts of models. *The mean reverting characteristics imply that there ought to be easily predictable trading strategies that investment managers can follow which are highly likely to make large profits at low risk.*

According to the Wilkie model, if at some point in time yields are very high for one asset category and very low for another, then over the following year the second category should on average significantly outperform the first. Suppose we adopt a mechanical rule following this approach, always switching at the end of each year to the asset category most likely to perform best (according to the model). Smith (1996) demonstrates that such a strategy will on average over the next 50 years outperform even the best performing asset category by about 5% pa! This seems to be much too high to be credible.

5.11 Would such relatively easy to predict strategies actually make these kinds of excess returns in practice? I think not. It is sometimes claimed that the Wilkie model detects persistent inefficiencies in markets, and that it picks these up correctly. It is perhaps more reasonable to suppose that, if such inefficiencies existed in the past, they should now have been 'identified' and should disappear in the future, being arbitraged away. Tomorrow's inefficiencies, if they exist, are likely to spring up in new directions.

5.12 The key problem with such model behaviour is that if we are testing the merits of dynamic investment strategies or ones involving derivatives it becomes very difficult to tell whether the analysis is picking up the fundamental advantages or disadvantages of the strategy or merely these anomalies.

6. PRICING DERIVATIVES IN THE WILKIE MODEL FRAMEWORK

6.1 The key constraint for a model not to exhibit such anomalies is for it to be *efficient*. This term is often loosely bracketed with the concept of *arbitrage-free*. Indeed the terms are often used interchangeably in market parlance.

However, strictly speaking an economic model exhibits arbitrage only if *risk-less* profits can be made, i.e. there are trading strategies guaranteed to produce a positive profit some of the time and never a negative profit, without any initial outlay of capital.

The Wilkie model is arbitrage-free in this sense since it has insufficient components to permit the identification of a completely risk-less arbitrage strategy. However, it is inefficient in the sense that it permits trading strategies which exhibit anomalously high profits for the amount of risk being undertaken.

6.2 The fact that the Wilkie model framework is arbitrage-free means that it is possible to price derivative securities using no-arbitrage type arguments within it.

6.3 In an arbitrage-free framework, and in the absence of transaction costs, then Kemp (1996) shows that the price of a vanilla put (or call) option is given by a suitable generalisation of the Black-Scholes option pricing formula as long as:

(a) the volatility of the price of the asset underlying the option, or to be more precise its *cumulative quadratic variation*, is known in advance; and

(b) the price does not jump, i.e. the cumulative quadratic variation is continuous.

6.4 The Wilkie model, because it uses normally distributed random components, does not exhibit jumps, as long as the random components are deemed to accrue continuously throughout each year.

6.5 The key step in identifying the price of such options is to identify the cumulative quadratic variation of the assets underlying the option. I have set out in Appendix C how this can be done, together with corresponding approximate pricing formulae for 'total return' options (i.e. ones with pay-offs linked to the differences between the total returns on different asset sectors).

6.6 A basic concept within derivative pricing in a no arbitrage framework is the *risk-neutral probability distribution*. As long as there is such a distribution, the fair price of any derivative (and indeed any sort of contingent claim) is the discounted expected value of the pay-off of the derivative *assuming that the price movements follow the risk-neutral probability distribution*. Different investors may assign different likelihoods to the same events, but all we need to know to identify fair prices for payments linked to such events are their risk-neutral probabilities.

6.7 Appendix C shows that in the Wilkie model the risk-neutral log returns $\boldsymbol{r}$ on different asset categories over n years follow an approximately multi-variate normal distribution $N(\boldsymbol{R}(n), nS^2)$, where S is the covariance matrix for one yearly returns, and $\boldsymbol{R}(n)$ is a vector which is the same for each asset class i, but potentially varying according to the length of the period under analysis. The standard deviation $\sigma(n)$ of the geometric average annualised 'risk-neutral' return on a particular asset category over n years is thus approximately $\sigma(1)/\sqrt{n}$.

6.8 The likelihoods of 'actual' outcomes defined by the Wilkie model have significantly different characteristics. Mean returns vary by asset category, but this is usual with any stochastic asset model, to capture users' views concerning differences in the long-term returns available on different asset categories.

More importantly, the mean reverting characteristics of the Wilkie model mean that the distribution of 'actual' returns over periods of time greater than one year is more bunched than the risk-neutral distribution.

For example, Wilkie (1995) uses Monte Carlo simulations to estimate the standard deviations of *nominal* total returns on UK equities (although admittedly not of log returns) geometrically averaged over various periods. His estimates (using the model's 'neutral' parameters) are given in the first row of the table, whereas the corresponding variabilities if there were no mean reverting tendencies in the model are given in the second row.

Term (n) in years	1	2	5	10	20	50
1. $\sigma^{actual}(n)$	19.47	12.71	7.41	4.80	3.48	2.31
c.f. 2. $\sigma^{actual}(1)/\sqrt{n}$	19.47	13.77	8.71	6.16	4.35	2.75
1. as a % of 2.	100%	92%	85%	78%	80%	84%

The equivalent comparison for *real* returns (in excess of the RPI) on UK equities, also based on simulations in Wilkie (1995), shows an even greater disparity:

Term (n) in years	1	2	5	10	20	50
1. $\sigma^{actual}(n)$	20.25	13.06	7.41	4.57	2.89	1.73
c.f. 2. $\sigma^{actual}(1)/\sqrt{n}$	20.25	14.32	9.06	6.40	4.53	2.86
1. as a % of 2.	100%	91%	82%	71%	64%	60%

6.9 Another difference between the risk-neutral and the 'actual' probability distribution of the Wilkie model is that the Wilkie model risk-neutral correlations between different asset categories are roughly constant over whatever time period is being analysed, but the correlations for 'actual' outcomes are not, as is again illustrated by the results of simulations in Wilkie (1995):

Term (n) in years	1	2	5	10	20	50
correlation between real returns on equities and gilts	0.47	0.44	0.38	0.36	0.25	0.09

6.10 With any model incorporating views about the likely expected performances of different asset categories, it is necessary to take into account the riskiness of different assets.

With the Wilkie model (or other mean reverting models), it is *also* necessary to bear in mind that the spread of outcomes revealed by simulations needs adjustment before it provides an adequate assessment of the 'price' associated with these risks.

7. AN ALTERNATIVE STOCHASTIC INVESTMENT MODEL

7.1 If we decide that mean reverting models like the Wilkie model are inappropriate for asset/liability modelling involving dynamic investment strategies then we need to come up with an alternative. It seems to me that the ideal economic model should be:

(a) consistent with both past history and economic intuition (insofar as these are not inconsistent with each other);

(b) relatively simple to understand (yet sufficiently comprehensive to permit the implications of different investment strategies to be tested); and

(c) efficient (and arbitrage free).

7.2 Requirement (c) is quite restrictive, as it essentially requires that the components of the model relating to asset returns cannot be mean reverting. Other components of the economic model which have no specific underlying asset are not constrained in the same way. Hence, we could probably still use the Wilkie approach for RPI inflation. We could also use the Wilkie wage inflation model, but it may be simpler merely to assume that real wage growth is normally distributed, with a relatively modest annualised standard deviation and a mean of, say, around 1.5-2.5% pa.

7.3 A simple way of achieving all these requirements is to assume that returns in different periods are independent and that they are distributed according to a multivariate log-normal distribution. This means that the (log) returns on each asset category in isolation follow a normal distribution with some mean and variance, and that there is some pre-defined correlation between the returns on any two different asset categories. Indeed, this is similar to the form of the Wilkie risk-neutral probability distribution, except that the mean returns are allowed to vary by asset category.

7.4 A potentially significant advantage of this alternative model is that optimal dynamic investment strategies under it may be categorised relatively simply (see Appendix B.3), in contrast to the much more complex categorisations applicable to mean reverting models.

7.5 The risk-neutral and 'actual' probability distributions of this model are the same (apart from the risk-neutral probability distribution having the same mean return on all assets). Thus there is no need as in Section 6 to adjust the spread of simulated outcomes before it is possible to put a 'price' on the riskiness of different asset classes.

7.6 It would probably be appropriate to concentrate on real returns (i.e. returns in excess of inflation) when defining the model. Most pension scheme liabilities are 'real' in nature, and therefore it is the real returns on assets that are particularly important not their nominal returns. However, it is also possible to specify such a model in terms of nominal returns.

Mixed specifications, which focus on real returns for some asset categories and nominal returns for others could also be adopted. For example, we might argue that the nominal returns on monetary assets such as fixed interest securities and cash should be largely independent of inflation, and therefore that the real returns on such assets should be negatively correlated to inflation. However, some care is needed when using mixed specifications whilst also using an a mean reverting model, like the Wilkie one, for future RPI. If the mixed specification includes varying correlation coefficients between the RPI itself and the returns on different asset categories then mean reverting characteristics slip back into the asset components of the model. Better perhaps would be to allow nominal returns on monetary assets to be correlated to the random component driving the RPI as this does not re-introduce mean reverting elements to the asset components of the model.

7.7 I noted in Section 5.7 that lognormal distributions might be inappropriate for return series. JP Morgan's RiskMetrics™ technical manual (1995) notes that distributions of returns seen in practice seem to have the following characteristics:

(a) The distributions have fat tails (i.e. there are more occurrences far away from the mean than if the distributions were normal), and the peak around the mean is higher than that predicted by the normal distribution. In statistical terminology, the distributions have a kurtosis greater than that of the normal distribution;

(b) The distributions may be negatively skewed (i.e. there may be more large negative returns than there are large positive ones); and

(c) Squared asset returns (i.e. volatility) often exhibit significant autocorrelation.

7.8 There are two main ways in which we can incorporate non-normality of returns within a stochastic investment model:

(a) We can retain the assumption that returns in different time periods are independently distributed, with parameters that do not depend on what has happened in the past, but use distributions with fatter tails than the normal distribution. These can accommodate the characteristics in section 7.7(a) and (b), but not (c). Two possible distributions examined by A.D.Smith (1995) that may be suitable are:

- Levy stable (otherwise known as stable Paretian) distributions. These have characteristics which make them difficult to handle mathematically, e.g. they have infinite variance (except for the normal distribution, which is a special case of the distribution). However, they also have some intuitively appealing characteristics, e.g. they have similar characteristics to fractals and other processes that seem to appear in real life. F. Longuin (1993) has analysed the returns on US equities. He concludes that they are fat tailed, but not as fat tailed as would be the case if the distributions were Levy stable.

- Differences of two gamma processes (one for up jumps and one for down jumps). This is Smith's preferred option. These distributions have sufficient parameters to enable any feasible skewness, kurtosis, variance and mean to be accommodated. They are less fat tailed than Levy stable distributions.

Another model seen in practice is the jump diffusion one, in which a jump process is explicitly superimposed on a random walk process arising from the use of normally distributed random components.

(b) Alternatively, models may use distributions whose parameters are conditional on what has gone before, e.g. GARCH type models and other models exhibiting stochastic volatility. These seem to be fashionable at present amongst researchers. They can accommodate the characteristics noted in Section 7.7(c) as well as those in Sections 7.7(a) and (b).

7.9 Asset/liability modellers need to decide how important it is to model the magnitude and incidence of outliers, and to adjust their models accordingly. One of the disadvantages of incorporating the sorts of adjustments just mentioned is that it considerably increases the complexity of the model. For example, if we extend the model in section 7.3 to include jumps, using gamma processes, then we might need perhaps double the number of parameters. Of course, the model would then also have a richer economic structure, which might make the extra complexity worthwhile in some circumstances.

It is not clear to me that such adjustments would always have much practical impact. If, say, decisions are taken on the basis of what constitutes moderately unlikely outcomes (e.g. with a 1 in 5 or a 1 in 10 chance of happening) then the potential size of extremely unlikely outcomes (e.g. ones with a 1 in 100 or 1 in 1000 chance of happening) may not be accorded great importance. A further complexity is that there is flexibility available to the Regulator to suspend the application of the MFR in "extreme circumstances". It seems to me that this makes accurate modelling of outliers impractical insofar as they affect the MFR.

7.10 The one aspect we have not yet addressed is the parameters to use in the model. If we adopt the log-normal approach suggested above, without trying to incorporate any element of stochastic volatility then we only need to identify suitable assumptions regarding:

(a) mean (real) returns for the various asset categories being modelled;
(b) variabilities of these returns; and
(c) the correlations between the returns on different asset categories (and with any other factors relevant to the behaviour of the liabilities).

These sorts of parameters are very much easier for pension scheme trustees to understand than those required for the Wilkie model.

7.11 As in Section 5.6(d), we have to face the issue of what we base our choice of parameters on. As hinted there, I prefer an approach in which the assumptions adopted are an amalgam of:

(a) past history;
(b) expectations for the future, e.g. based on surveys of investment managers' views; and
(c) common sense, bearing in mind the views of the consultant (and of trustee representatives) on the relative attractiveness of different asset categories.

These would generally provide parameters deemed to apply were the markets to be 'neutrally' priced at present. If a given market is temporarily cheap or dear, we might expect this to be picked up by the fund manager responsible for the scheme's asset allocation.

7.12 Possible assumptions for log returns, based on such an amalgam, adjusting property volatility upwards along the lines implied by e.g. Blundell (1996) might be:

	Mean Real Return % pa	Standard Deviation % pa	Correlations						
			Cash	Fixed Interest	ILG	Property	UK Equities	Overseas Equities	Overseas Bonds
Cash	1-2	3-5	1						
Fixed Interest	3-4	12-15	0.30	1					
ILG	3-4	5-7	0.20	0.50	1				
Property	4-5	15-20	0.20	0.10	0.10	1			
UK Equities	c.6	18-22	0.10	0.60	0.40	0.20	1		
Overseas Equities	c.6	18-25	0.25	0.25	0.25	0.20	0.55	1	
Overseas Bonds	3-4	12-16	0.20	0.40	0.45	0.00	0.60	0.70	1

8. VALUING ASSETS AND LIABILITIES

8.1 There is one further component of the economic model that we have not yet considered, namely the yields on different assets, e.g. the dividend yield on equities. A gross investor such as a pension fund should be essentially indifferent to whether investment return appears in the form of income rather than capital gains. Why then is the dividend yield important?

8.2 The reason is that such yields often appear in pension fund actuarial valuation calculations. In their ongoing valuations, actuaries have traditionally not valued assets at market value. Instead they have traditionally used some 'actuarially assessed' value of assets. These are designed, it is usually claimed, to smooth out over the medium term fluctuations in market values, so as to lead to a more stable company contribution rate being revealed at each regular ongoing actuarial valuation.

8.3 One way of carrying out this smoothing is to use some form of moving average of market values, e.g. as suggested by Dyson & Exley (1995). No assumption regarding running yields is needed with this approach.

8.4 However, a more common approach is to use a dividend discount model, which for UK equities effectively ends up valuing assets according to the following formula:

$$\text{Actuarial Value}(AV) = \text{Market Value}(MV) \times \frac{\text{current dividend yield}}{\text{neutral dividend yield}}$$

So, if the current dividend yield is high (relative to some assumed fixed neutral level) we write up the value, whilst if it is low we write it down.

This sort of adjustment (or rather its reciprocal) is also incorporated in the MFR to value liabilities deemed to be 'equity-linked' in its calculation.

8.5 There is a potential problem with this approach. If the 'neutral' dividend yield chosen for this purpose is wrong then the projected dividend yield will over the long term trend to the wrong value. This is, of course, a problem of the valuation methodology rather than of any stochastic model used to analyse its impact.

It is not clear to me that there is any objectively correct long-term 'neutral' dividend yield assumption. Yield levels are dependent on the degree to which investors seek immediate cash flows from their investments, and on whether these cash flows come from share buy-backs etc. or dividends. It is quite reasonable to postulate a wide range of 'normal' dividend yields without any of them fundamentally distorting total returns available to shareholders. Indeed 'normal' levels seem to vary by country. UK pension scheme actuaries have traditionally adjusted their 'neutral' dividend yield assumptions when they seemed to have become divorced from reality. It will be interesting to see how this problem is tackled within the MFR. It is by no means obvious how best to model such changes (which are likely to be step changes).

It is also not entirely clear to me that the dividend discount method will always lead to a smoother progression of values. It seems to work much better when applied to UK equities than to, say, US equities.

8.6 Whatever the merits of a dividend discount method, we will need some way of modelling how the UK equity dividend yield may vary, if only for the purpose of the MFR. One solution would be to use the approach in the Wilkie model. However, this would then reintroduce mean reverting characteristics into our model. An approach which is more robust is to:

(a) determine the total return (on market values) in some arbitrage free way;

(b) determine a suitable way of placing an 'actuarial' value on the assets with the required smoothing characteristics, e.g. the Dyson & Exley approach; and

(c) back-out the implied dividend yield by rearranging the equation in Section 8.4 to be:

$$\text{projected dividend yield} = \text{neutral dividend yield} \times \frac{AV}{MV}$$

8.7 Dyson & Exley indicate that the probability characteristics of the dividend discount model can be replicated by a suitable moving average. Therefore, such an approach ought to be able to achieve the same sorts of spreads of outcomes as would be revealed using, say, the Wilkie model.

8.8 We might also add in an additional random component to reflect the fact that dividend yields may be influenced disproportionately by some factors, like changes in dividend retention levels, advance corporation tax rates etc., which do not have as large an impact on market values. The same sort of approach can, if necessary, also be applied to overseas equities and property.

8.9 We may also need to make assumptions about the yields on other assets, principally those on fixed interest and index-linked securities. Take for example the real gross redemption yield on index-linked gilts. In actuarial valuation calculations this might appear in an adjustment of the following form:

$$\text{AV of ILGs} = \text{MV of ILG's} \times \frac{(1+j)^{-n}}{(1+i)^{-n}}$$

where j = long-term 'neutral' real rate of return on ILG's, i = current gross redemption yield and n = duration of the index-linked gilts. The above formula is a simplified version of what would be used in practice, as it assumes that the index-linked gilt is a zero coupon bond. It also ignores the eight month time lag between the relevant date used for indexing, and the date of the relevant asset payment.

8.10 The logic behind this adjustment is as follows. Suppose that a fixed real rate of interest j is used in future actuarial valuations. A liability which is a commitment to pay in n years time a sum of £1 (inflation linked between now and then) will then be valued as $(1+j)^{-n}$. It will therefore grow at a constant rate $(1+j)$, in real terms, whatever happens to current market conditions. If we invest in the matching asset and value that asset in the way described above, then its 'actuarial value' should also grow at the same constant rate in real terms, whatever happens to current market conditions.

8.11 If the liabilities and matching assets were this simple then we can identify precisely how the gross redemption yield $i(t)$ needs to vary to be consistent with the modelling of the market value of index-linked gilts. If t is the time from outset, and n is the duration of the notional zero coupon ILG at outset then:

$$AV_t = MV_t \frac{(1+j)^{-(n-t)}}{(1+i(t))^{-(n-t)}}$$

$$\Rightarrow \quad \log(AV_t) = \log(MV_t) + (n-t)\log(1+i_t) - (n-t)\log(1+j)$$

$$\text{and} \quad \log(AV_{t+1}) = \log(MV_{t+1}) + (n-t-1)\log(1+i_{t+1}) - (n-t-1)\log(1+j)$$

$$\text{but} \quad AV_{t+1} = (1+j)AV_t$$

$$\Rightarrow \quad \log(1+i_{t+1}) = \frac{n-t}{n-t-1}\log(1+i_t) - \frac{1}{n-t-1}\log\left(\frac{MV_{t+1}}{MV_t}\right)$$

8.12 In these circumstances, the outstanding duration of the gilt falls by 1 year as each year passes, and would eventually reach zero. The recurrence relationship is unstable, in the sense that $i(t)$ tends to $\pm\infty$, unless j is chosen at outset to match the then gross redemption yield of the notional index-linked gilt. Again this is a problem introduced by the valuation methodology rather than one linked to asset/liability modelling per se.

8.13 In reality, life is more complex than set out in Section 8.11. The overall duration of the market index and of the corresponding liabilities typically gets replenished and remains roughly constant (ignoring changes that occur because of changes in yields). Hence, we could adopt a more complicated modelling approach such as:

$$\log(1+i_{t+1}) = a.\left(\log(1+r) - \log(1+i_t\right) + b.\log(1+i_t) - \frac{1}{c}.\log\left(\frac{MV_{t+1}}{MV_t}\right) + \varepsilon$$

where r is some long-term 'neutral' yield, which the yield tends to revert to over time, a defines some tendency to tend back to this level, b replaces $(n\text{-}t)/(n\text{-}t\text{-}1)$, c is a constant linking the yield to the return on the index (probably close in value to the duration of the index or the liabilities) and ε is some additional random component, e.g. a normally distributed random variable with some suitable standard deviation. b needs to be less than 1 in this formula to avoid long term instability in the recurrence relationship.

8.14 Exactly the same logic can be applied to fixed interest gilts, except that the parameters and returns may need to be re-expressed in nominal rather real terms.

8.15 If we wish to be more sophisticated whilst still retaining our desire for an arbitrage-free model, then we could introduce a term structure approach, modelling the evolution over time of the entire yield curve (i.e. how yields vary by duration). This sort of approach is used in practice by derivatives specialists in investment banks. It can in principle be applied to any asset category.

8.16 There is one final important issue that arises in practice when valuing assets (or sometimes liabilities). When actuaries value the assets using the sorts of formulae described above, they do not necessarily use the actual asset mix adopted by the pension fund. Instead, they typically value the assets by notionally reinvesting them in some portfolio. The idea is that the notional portfolio in some sense reflects or is 'suitable' for the liabilities.

The key problem this presents in asset/liability modelling is that we may be potentially testing very different investment strategies within the same study. We therefore need some rule that will identify how the notional portfolio used will vary if the long-term investment strategy is altered. There are two main schools of thought:

(a) If we have concluded that some benchmark is appropriate for the scheme (especially if we have carried out some form of asset/liability study), then perhaps this benchmark should also form the notional portfolio used for valuation purposes.

(b) Alternatively, the notional portfolio might always relate to a portfolio that can be said to 'match' the liabilities, e.g. appropriate gilts or ILG's for pensioners, etc.

8.17 These two approaches treat differently the impact of experience differing from assumptions. For example, if the assets grow faster (or slower) than expected relative to the liabilities, then approach (b) will reflect the whole of these experience gains or losses at the first actuarial valuation after they arise. The MFR effectively uses this approach, with younger actives' liabilities being deemed to be matched by equities. In contrast, approach (a) will typically smooth out such ups and downs, leading to less volatile contribution rates being revealed, but also potentially understating the impact of extreme gains or losses.

8.18 The two approaches also have a different impact on the opening valuation results. These will in theory depend on the actual investment strategy adopted for method (a) but not method (b).

Actuaries and others intimately aware of the calculations involved may not be uncomfortable with valuation results that can depend on the investment strategy being adopted. Lay trustees may however be less convinced that this should be the case, since it seems to contradict common sense (unless explicitly justified using some kind of mismatching reserve). Otherwise, it would seem possible to improve the funding position of the scheme merely by shifting investment strategy immediately before the valuation, and reversing the shift immediately afterwards. The MFR, because it adopts method (b), is not open to manipulation in this manner.

8.19 This section has highlighted several problems which result from the valuation of assets and/or liabilities at other than market values. Exley & Mehta (1996) describe off-market bases in actuarial valuations as "*arbitrary accounting conventions (akin to book value accounting)*". They warn that these bases "*may give the appearance of spurious return enhancement or risk reduction for certain strategies in violation of the reality which we would see if we looked through to the underlying economic position on the market-value related basis*".

It would certainly make life much easier for asset/liability modellers if assets and liabilities were always valued at market value, as is generally done for asset/liability modelling within banks.

9. CONCLUSIONS

9.1 The Wilkie model implies the existence of market anomalies which lead to abnormally high profits.

9.2 The mean reverting characteristics that give rise to these anomalies also result in a strange relationship between the spread of 'actual' outcomes postulated by the model and its risk-neutral probability distribution on which the prices of derivatives would depend. In particular:

(a) The spread of 'actual' outcomes over longer periods is significantly narrower than the risk-neutral probability distribution; and

(b) There can be material differences in the correlations between long-term returns on different asset categories.

9.3 Simplistic use of the Wilkie model (or other mean reverting models) without allowance for these effects may therefore misstate the value of options and produce misleading conclusions when testing dynamic investment strategies.

9.4 A more robust and easier to understand model assumes that returns in individual years are independent multivariate log-normally distributed. Its risk-neutral probability distribution is the same as its 'actual' probability distribution, except for differences in the long-term returns on different asset categories. It is thus easier for users to understand and correctly quantify the basic trade-off between risk and return for the various asset categories.

With this model, optimal *dynamic* investment strategies are also relatively easy to categorise, which is not the case for mean reverting models.

9.5 The widespread use of the dividend discount method for valuing assets in ongoing actuarial valuations, and the introduction of an equivalent process in the MFR means that it is necessary to include some method of projecting dividend yields in pension fund stochastic investment models.

However, there are some inherent problems with this sort of off-market valuation methodology which considerably complicate the asset/liability modelling process. For example, it is not clear why there should be any fundamentally stable long-term 'neutral' dividend yield to which equity yields can be expected to trend, even if one has been adopted for the MFR. This means that at some stage in the future there may be step changes in the MFR basis, but when these might occur is virtually impossible to predict.

It would be much easier from an asset/liability modelling perspective if assets and liabilities were always valued using market values (or estimates of market values). If smoothing of market values is deemed necessary then the use of moving averages seems more robust, from an asset/liability modelling perspective, than the use of dividend discount adjustments.

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APPENDIX A

IMPLEMENTING ASSET/LIABILITY MODELLING

A.1 Carrying out the projections in an asset/liability study requires a computer model of how the scheme might behave in the future. Typically, a simplified model of the pension scheme is adopted, e.g. grouping individual members by age and aggregating similar membership categories together. Members who are currently active, members who are now deferred pensioners and members who are in receipt of pensions would normally be distinguished. The development of the liabilities attributable each category is very different.

A.2 This simplified model would normally be constructed so that it:

(a) closely approximates to the actual characteristics of the scheme; and

(b) permits without excessive difficulty the calculation of all the key characteristics of the scheme likely to be of interest, not only at the start of the projection but also later on.

Some specific allowance for new entrants would typically be made. This contrasts with the actuarial valuation, which typically only takes new entrants into account in an implicit fashion.

A.3 It is usually considered helpful if the model:

(a) groups data by age for at least each of the main membership types (i.e. actives, deferreds, pensioners and dependants);

(b) is calibrated so that it exactly replicates as its starting position the results of the latest actuarial valuation; and

(c) operates in such a way that it can project (with a high degree of accuracy) not only *future cash flows* but also *future valuation results* (and any related features such as the MFR funding levels or SSAP 24 pension costs that are being analysed). Ideally, the assets and liabilities at the relevant point in time should be able to be valued on any arbitrary *valuation* basis. The model also needs to be able to permit accurate projection of the assets and liabilities in the meantime on any arbitrary *projection* basis.

A.4 The main items of data that need to be collated for such a model are:

(a) The results of the latest ongoing actuarial valuation (up-dated if it is now out of date), to calibrate the asset/liability model;

(b) The results of any calculations of the current position regarding the MFR, scheme-specific measures of solvency and accounting costs etc., if these are to be analysed in the study. Backing calculations and methodology would be needed if the calculations are complex, or if there are various ways in which the calculations could be carried out;

(c) Details of the benefit structure of the scheme. The details contained in an actuarial valuation report are usually sufficient for this purpose;

(d) Membership liability data grouped by age for each of the main membership categories. For pensioners (and dependants) this would be current pensions. If parts of the pensions received different levels of increase (e.g. GMP and non-GMP elements) then further information on the breakdown would usually be needed. For deferred pensioners the data would relate to current (or projected) deferred pensions. For active members the data would be accrued pension and, separately, current pensionable salary. Again, for deferred pensioners and active members, it may be necessary to subdivide the data according to the characteristics of the liabilities.

If data grouped by age is not available then average ages or other aggregate characteristics of the liabilities would typically be sought.

(e) Some indication of trends in new entrants; and

(f) Details of current investment policy (and probably the current investment management structure), to identify what are sensible investment strategies to concentrate on in the study.

A.5 In addition to assumptions which make tractable the projections, it is also necessary to make assumptions regarding:

(a) *investment policy*, i.e. the choice of investment policy to analyse and whether they should be limited merely to static ones or also include dynamic ones;

(b) *economic*, i.e. the stochastic investment model underlying the projections;

(c) *methodology*, e.g. what constitutes solvency and what key features need to be projected;

(d) *demographic*, e.g. what are suitable assumptions for retirements and deaths; and

(e) *funding and valuation policy,* e.g. how contribution rates will progress in the future, and how ongoing actuarial valuations will be carried out.

The first two, i.e. the extension of investment strategies to include dynamic ones and the choice of stochastic investment model, are the main topics of this paper. The remaining sorts of assumptions are discussed more briefly below:

A.6 *Methodology*

The first requirement is to decide which key features to project, such as ongoing funding levels, solvency levels and company contribution rates (see Section 3.1).

Nowadays, 'solvency' would normally be the most important feature being analysed. Typically it would now be measured in a manner consistent with the MFR (even though this is not strictly speaking a measure of solvency), but sometimes other more scheme-specific definitions of solvency may be more appropriate.

Allowance would normally be made in the projections (although not necessarily in future valuations) for discretionary pension increases if there is an expectation that they will occur in the future. Usually the consultant would assume that no other benefit improvements are awarded during the projection period. However, this may be varied if the trustees and the sponsoring company have other plans.

A.7 *Demographic Assumptions*

The projections typically start as at the date of the latest actuarial valuation. If this was some time ago, the valuation may be updated approximately to a more recent date. The preferred timespan of the projections seems to vary. Arguably, anything much longer than, say, 10 years, is subject to so much uncertainty as to be of limited benefit. However, some trustees ask for projections for the next 20 years or even longer.

The projections would usually adopt the same demographic assumptions in the projections as used in the latest actuarial valuation.

The default approach to new entrant numbers would typically be to assume stability (either of active membership numbers or of pay-roll in inflation adjusted terms). However, the sponsoring employer may be aware of a significant increase or decrease in expected numbers. For example, there may be a planned acquisition or redundancy exercise. These would then be taken into account as appropriate. Sensitivity may be required in phrasing wording describing such exercises in any formal reports that are prepared.

A.8 *Funding and Valuation Policy*

Funding policy is the generic term applied to the process of deciding the speed at which contributions are paid into the pension scheme. It will depend on two things:

(a) *The actuarial valuation basis being used.*

If this is 'strong', i.e. prudent, then the value placed on liabilities will be relatively high and relatively high contributions will be requested from the employer. If actual experience turns out to be more rosy than assumed, the contributions can be expected to fall over time. Conversely, if the basis is 'weak', then lower contributions will be required from the employer at outset, but the contributions will not fall by as much over time, or may increase. The key point is that it is the *actual* experience (and not what is assumed) that ultimately determines the *actual* long-term cost. The valuation basis influences only the speed at which this cost is accrued.

(b) *What happens if a surplus or deficit is revealed at an actuarial valuation.*

If a surplus is revealed and the company is short of cash then it may seek a contribution holiday over a relatively short period of time. In extreme circumstances, it may even seek a refund of the surplus. Conversely, if there is a deficit, the company may be keen to spread it over many years (but the trustees may wish it to be made good quite quickly). The position is further complicated because the accounting cost charged in the company accounts may not equal the actual cash payments into the scheme.

There is therefore considerable flexibility in what contributions will be paid in the future. It will depend on the views of the trustees, the actuary and, to some extent, the employer at each future actuarial valuation. Extra contributions under the MFR may also arise.

In practice, these factors would be modelled in some relatively simple fashion, e.g.:

(a) The *ongoing* actuarial valuation basis might be assumed to remain constant throughout the projection;

(b) If at any future valuation there is a surplus (or deficit) of assets over past service liabilities, then this surplus (or deficit) is assumed to be spread over the average future working lifetime of active members as a level percentage of pensionable salary; and

(c) Any extra contributions under the MFR are paid as required by the MFR.

Several different funding strategies may be analysed, so that the trustees can understand the implications of different approaches.

The valuation bases for measures like MFR funding levels or scheme-specific solvency would generally be assumed to vary according to the then current market conditions, as projected by the stochastic investment model. In particular, they will often be heavily influenced by the projected yields on gilts (and for the MFR, equity dividend yields).

APPENDIX B

PORTFOLIO OPTIMISATION AND EFFICIENT FRONTIERS

B.1 Portfolio Optimisation

B.1.1 The conventional method used to identify 'optimum' portfolios involves defining:

(a) a measure of 'reward' or 'return' from each asset category
(b) a measure of 'risk' from each asset category, and
(c) correlations between different asset categories, so that the returns and risks of *mixtures* of assets can be identified.

Constraints are also placed on portfolios. Typically portfolios are required not to be geared or short-sold in any market. Constant asset mixes are usually assumed (but see Appendix B.3).

Any acceptable portfolio can then be categorised by its 'return' and 'risk', measured as per (a) and (b). For any given level of return there will be a portfolio (or occasionally more than one portfolio) with the minimum possible risk. Or, equivalently, for any given level of risk there will be a portfolio which has the greatest possible return. Portfolios with these characteristics are described as *efficient* portfolios. These will range from one with the lowest possible risk to one with the highest possible return (if as is usual our measures of risk and return are such that increasing risk is rewarded with extra expected return).

The whole series of efficient portfolios each achieving an optimal trade off between risk and reward (for some given risk) is known as the *efficient frontier.*

B.1.2 'Reward'/'return' in such an analysis could be measured in terms of expected log returns, which thus compound geometrically (although some practitioners use expected returns instead).

B.1.3 Agreeing a suitable measure for 'risk' is more difficult. The original measure of 'risk' used in such analyses was the standard deviation of the (nominal) return on the portfolio. However, this only relates to 'risk' in terms of potential falls in capital values. For pension funds, a better alternative is to measure risk by reference to the liabilities, typically taking as a proxy for 'risk' the standard deviation of return relative to the portfolio which most closely 'matches' the liabilities. There may be one sort of matching portfolio (and hence risk) appropriate to ongoing funding and another for the MFR.

B.1.4 Alternatively, more sophisticated measures of risk may be adopted, e.g. ones based on:

(a) failure to meet some specific liability objective; and/or

(b) the likelihood of achieving a set target outcome at some future time horizon.

Non-symmetric measures of risk, e.g. semi-variance may also be used. In general, all risk measures assign some *penalty function* dependent on outcomes and then determine an aggregate risk measure which is some function of the average of this penalty function. For example, standard deviations are square roots of a quadratic penalty function.

B.1.5 Portfolios that are 'efficient' if risk is measured using one risk measure will also be efficient if risk is measured in any other way which is a monotonically increasing function of the first. Thus efficient portfolios based on using standard deviations are also efficient if risk is measured using variances (since variance is the square of the standard deviation). The technique described in Section B.1.3 is therefore often called mean-variance optimisation.

Efficient portfolios for different sorts of risk measures are often surprisingly similar, as long as the underlying type of risk being measured, i.e. the underlying sort of objective involved, is similar. Indeed for some stochastic models they can be identical for any risk measure that might be chosen, as long as it relates to the same sort of overall objective (see Appendix B.3.3).

B.2 Finding the Mean-Variance Efficient Frontier

B.2.1 A common way of defining efficient frontiers is the 'one period' approach described below. It has similarities to the approach proposed by Wise (1987).

Suppose that there are n asset categories and that a portfolio has proportions $\boldsymbol{a} = (a_1,\ldots, a_n)$ in each, with $\Sigma a_i = 1$. Suppose the 'expected' rewards on these categories is $\boldsymbol{r} = (r_1,\ldots, r_n)$. The expected reward of mix $\boldsymbol{a}$ can then be defined as $r(\boldsymbol{a}) = \boldsymbol{a}.\boldsymbol{r} = \Sigma\, a_i.r_i$.

Similarly, let $\boldsymbol{s} = (s_1,\ldots, s_n)$ where s_i. is the 'uncertainty' of asset a_i in isolation. It is also necessary to allow for the correlation between different asset categories, which we could do by identifying the correlation c_{ij} between category a_i and category a_j. These correlation coefficients would lie between -1 and +1 with $c_{ii} \equiv 1$ for all i.

Suppose we can identify a *matched* or minimum risk portfolio consists of $\boldsymbol{m} = (m_1,\ldots, m_n)$ where again $\Sigma m_i = 1$.

There will be some non-investment related features which cannot be protected against by investment policy. We shall assume that the standard deviation of this non-investment related or extraneous risk is e, and that it is uncorrelated with investment risk.

B.2.2 Then the uncertainty or 'risk' of portfolio $\boldsymbol{a}$ relative to portfolio $\boldsymbol{m}$ would be defined as $S(\boldsymbol{a},\boldsymbol{m})$ where:

$$S^2 = e^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - m_i)v_{ij}(a_j - m_j) = e^2 + (a-m)^T V(a-m) \quad \text{where } v_{ij} = s_i c_{ij} s_j$$

The term in e^2 does not affect the portfolios that are deemed 'efficient'. The part that is investment related is nil for the minimum risk portfolio, $\boldsymbol{a} = \boldsymbol{m}$. The larger the deviation away from the matched position, the larger is the risk *relative* to the low risk portfolio. The more the deviation is concentrated towards unpredictable categories (i.e. categories with large s_i) and the less there are compensating correlations between asset categories then again the larger is the risk relative to the minimum risk portfolio.

B.2.3 The efficient frontier is the set of asset mixes which trade-off reward and uncertainty most efficiently (i.e. no asset mix has a higher reward for a given level of uncertainty). Hence the efficient frontier relates to the risk and return characteristics of all 'valid' portfolio mixes which maximise r for any specific S, or correspondingly minimise S for any specific r. Valid portfolios are ones whose proportions add to 1 and where each a_i are subject to the appropriate constraints. We can think of r as like a mean (of log returns) and S^2 as like a variance (of log returns), so this sort of efficient frontier would often be called a *mean-variance efficient frontier.*

This can be shown (using Lagrange's principle) to be equivalent to finding the portfolios which maximise G(k) where:

$$G = k.r - (1-k)S^2 \qquad (\text{where } 0 < k < 1)$$

B.2.4 If k is close to 0 the above formula places very little weight on achieving high 'expected' returns, i.e. such portfolios concentrate almost exclusively on minimising risk relative to the liabilities (and will be very close to the matched portfolio). If k is close to 1 the formula places very little weight on minimising risk and will consist of a portfolio concentrating very largely on those asset categories providing the highest 'expected' returns.

B.2.5 There are a number of standard mathematical algorithms that can be used to identify portfolios that lie on this efficient frontier. The way we have defined G(k) means that it is a quadratic function, and we are aiming to find the maximum value of it subject to constraints. Thus the problem is often known as *constrained quadratic optimisation.*

A standard approach described in several text-books is to make use of a modified version of the Simplex Algorithm, see e.g. Taha (1971). This algorithm only works if the quadratic expression being maximised is negative definite (otherwise there is no unique solution to the optimisation problem), but this should not normally be a problem in practice.

B.2.6 The portfolios deemed efficient can be very sensitive to the assumptions. There are usually a large number of portfolios all nearly efficient. This is because maximising $G(k)$ is, except where the constraints bite, equivalent to finding where the first partial differentials of $G(k)$ are zero. All nearby portfolios will therefore have only marginally lower values of $G(k)$. Attempts have been made to reduce the dependency of the results to the assumptions. Typically, these involve adjusting the assumptions so that the results for some level of risk are consistent with some asset mix derived from other principles. For example, we may require that the average asset mix of UK pension schemes is efficient for some level of risk.

B.2.7 Ignoring the term in e^2, the mean-variance efficient frontier described above is an arc consisting of a small number of straight lines, but always concave downwards (if the x axis is risk, S^2, and the y axis return, r). The kinks are where individual constraints on asset mixes (as per Appendix B.1.1) kick in.

If there are no such constraints then the mean-variance efficient frontier becomes a straight line and efficient portfolios take the form (much like in CAPM) of:

$$b \times \text{'matched' portfolio} + (1-b) \times \text{'risky' portfolio}$$

where the risky portfolio is the same for all efficient portfolios, and only b varies.

B.3 Dynamic Optimisation

B.3.1 Identifying dynamic investment strategies that are optimal seems intuitively much more complicated than identifying merely optimal static investment strategies. Smith (1996) describes several possible techniques that can be applied generally, but they all seem non-trivial.

B.3.2 Fortunately, if we constrain our stochastic investment model to be continuous, to have predefined normal random components (i.e. to be combinations of predefined diffusion processes) and we ignore transaction costs then dynamic optimisation becomes much simpler. This is because:

(a) Derivatives become perfectly replicable by dynamic investment strategies (and vice versa); and

(b) If we split the whole period under analysis into many short components, then the optimal dynamic investment strategy is one that involves rebalancing the portfolio at the start of each period to be whatever is optimal for that period in isolation, given the position then reached. The optimal position can thus be determined inductively, by working backwards from some ultimate horizon, in the same fashion as can be used to value derivatives, see e.g. Kemp (1996).

We already know how to optimise each short period in isolation. We merely use the constrained quadratic optimisation described in Section B.2.

B.3.3 Indeed, an important result to note is that if (log) returns in a one period optimisation are normally distributed then whatever the underlying measure of risk adopted, any portfolio which is efficient under that measure is also mean-variance optimal. The normal distribution has the characteristic that all its moments and hence its entire probability distribution are uniquely characterised by just its mean and standard deviation. Hence the average of any penalty function is merely a function of these two parameters. This simplification applies to both the Wilkie model and the alternative model developed in Section 7.

B.3.4 However, in the Wilkie model the expected returns on different asset classes over the immediate future and hence the structure of the 'risky' portfolio as per, say, Section B.2.7 are highly sensitive to the precise values of the state of the model at the point in time under analysis. This means that the optimal dynamic investment strategy under the Wilkie model will vary hugely as time progresses in a very complicated fashion. This is as we might have expected given the observations in Sections 5.10 – 5.12. The optimisation process is driven principally by the anomalies introduced by the mean reverting characteristics of the model.

B.3.5 In contrast, the model described in Section 7 does not exhibit this problem. In the absence of gearing constraints and as long as the matched portfolio $\boldsymbol{m}$ remains fixed, the optimal dynamic investment strategy is as given in B.2.7, with a constant 'risky' portfolio but with b varying dynamically (whatever the precise risk measure under consideration).

The dynamic optimisation problem then in effect simplifies to a two asset class problem, much like the benchmark approach described in Section 2.3(b), but with the proportions in the two composite asset 'classes' varying dynamically.

B.3.6 If jumps or stochastic volatility are introduced into the models, or if transaction costs are allowed for, then the optimisation problem once again becomes much more difficult. It is no longer preference independent and in general depends on the precise measure of risk being used, see Kemp (1996).

APPENDIX C

OPTION PRICING FORMULAE FOR THE WILKIE MODEL

C.1 We shall concentrate on 'total return' derivatives. An example would be a European-style put option which gives the holder the right to sell at time T a portfolio invested in an equity index, with gross income reinvested, represented by S_t, for a price set by reference to an initial exercise price, E, rolled up in line with the total return on cash. Suppose that F_t is the cash index with gross income reinvested. Then the pay-off of this option at time T is:

$$\max\left(E.\frac{F_T}{F_0} - S_T, 0\right)$$

The cash and equity indices can be replaced by any other asset categories being modelled.

C.2 The price of such an option at time t, in the absence of transaction costs, if the cumulative quadratic variation of the option, $C(t)$, is continuous and $C(T)$-$C(t)$ is fixed whatever the path taken by the underlying stochastic process, is given by the following formula, see Neuberger (1990) or Kemp (1996):

$$P(S,t) = E.N(-d_2) - S.N(-d_1)$$

$$\text{where } d_1 = \frac{\log(S/E) + (C(T) - C(t))/2}{\sqrt{C(T) - C(t)}}$$

$$d_2 = d_1 - \sqrt{C(T) - C(t)}$$

$$\text{and } N(x) = \text{cumulative normal distribution function}$$

We therefore need to identify formulae for the cumulative quadratic variation for such options within the Wilkie model framework.

C.3 The total return components of the Wilkie model develop according to the following formulae, where the total return indices at time t are: $PR(t)$ for equities, $CR(t)$ for consols (fixed interest gilts), $BR(t)$ for cash, $RR(t)$ for index-linked gilts and $AR(t)$ for property.

$$\begin{aligned}\log(PR(t)/PR(t-1)) &= DW \times DM(t) + (1-DW) \times I(t) \\ &+ DMU + DSD \times DZ(t) + \log(1 + \exp\{-YW \times I(t) - YN(t)\}) \\ &+ YW \times I(t-1) + YN(t-1) + DY \times YSD \times YZ(t-1) \\ &+ DB \times DSD \times DZ(t-1)\end{aligned}$$

$$\begin{aligned}\log(CR(t)/CR(t-1)) &= \log\left(1 + \frac{1}{CM(t) + CMU \times \exp(CN(t))}\right) \\ &+ \log(CM(t-1) + CMU \times \exp(CN(t-1)))\end{aligned}$$

$$\log\left(BR(t)/BR(t-1)\right)=$$
$$\log\left(1+\left\{CM(t-1)+CMU\times\exp\left(CN(t-1)\right)\right\}\times\exp\left(-BD(t-1)\right)\right)$$
$$\log\left(RR(t)/RR(t-1)\right)=I(t)+\log\left(1+\exp(-lnR(t)\right)+lnR(t-1)$$
$$\log\left(AR(t)/AR(t-1)\right)=\log\left(1+\exp(-lnZ(t)\right)$$
$$+EW\times EM(t)+(1-EW)\times I(t)+EMU+EBZ\times ZSD\times ZZ(t)$$
$$+ESD\times EZ(t)+lnZ(t-1)$$

C.4 In these formulae, the 'state' vectors are: $I(t)$ the rate of RPI inflation at time t, $YN(t)$ the element of the dividend yield not explained by prices, $CN(t)$ the element of long term interest rates not explained by retail prices, $BD(t)$ the relationship between long and short term interest rates, $lnZ(t)$ the log of the property yield, $lnR(t)$ the log of the real yield on index linked stocks and three 'quasi-state' vectors $DM(t)$, $CM(t)$ and $EM(t)$ which are moving averages of past RPI inflation. They develop as follows:

$$I(t)=QMU+QA\times\left(I(t-1)-QMU\right)+QSD\times QZ(t)$$
$$YN(t)=lnYMU+YA\times\left(YN(t-1)-lnYMU\right)+YSD\times YZ(t)$$
$$CN(t)=CA1\times CN(t-1)+CY\times YSD\times YZ(t)+CSD\times CZ(t)$$
$$BD(t)=BMU+BA\times\left(BD(t-1)-BMU\right)+BSD\times BZ(t)$$
$$lnZ(t)=lnZMU+ZA\times\left(lnZ(t-1)-lnZMU\right)+ZSD\times ZZ(t)$$
$$lnR(t)=lnRMU+RA\times\left(lnR(t-1)-lnRMU\right)+RBC\times CSD\times CZ(t)$$
$$+RSD\times RZ(t)$$
$$DM(t)=DD\times I(t)+(1-DD)\times DM(t-1)$$
$$CM(t)=CD\times I(t)+(1-CD)\times CM(t-1)$$
$$EM(t)=ED\times I(t)+(1-ED)\times EM(t-1)$$

C.5 The random components $YZ(t)$, $DZ(t)$, $ZZ(t)$, $QZ(t)$, $CZ(t)$, $BZ(t)$, $ZZ(t)$, $RZ(t)$ are independent identically distributed unit normal random variables, and are the source of the variability of the model. The remaining terms are constants chosen as parameters when the Wilkie model is first established by the user.

C.6 To first order, we can expand the right-hand sides of the formulae in the Section C.4 in the following form, for each asset category i:

$$y_i(t)=c_i+a_{i,YZ}YZ(t)+a_{i,DZ}DZ(t)+\ldots$$

where c_i and all the $a_{i,Z}$ are constants dependent only on the state variables at time t-1 or before.

C.7 The relevant values of the $a_{i,Z}$ are as follows:

Equities (i=1)

We note that:

$$DM(t) = DD * I(t) + (1-DD) * DM(t-1)$$
$$I(t) = QMU + QA * \left(I(t-1) - QMU\right) + QSD * QZ(t)$$
$$\log\left(1+e^{a+x}\right) \approx \log(1+e^{a}) + \left(\frac{e^{a}}{1+e^{a}}\right)x$$
and $YN(t) = lnYMU + YA * \left(YN(t-1) - lnYMU\right) + YSD * YZ(t)$

Hence we have, approximately:

$$\log\left(PR(t) / PR(t-1)\right) \approx c + (DW \times DD + 1 - DW - k \times YW) \times QSD \times QZ(t)$$
$$+ DSD \times DZ(t) \ - k \times YSD \times YZ(t)$$
where $a = -YW * (QMU + QA \times \left(I(t-1) - QMU\right) - lnYMU - YA \times \left(YN(t-1) - lnYMU\right)$

and $k = \dfrac{e^{a}}{1+e^{a}}$

If the model is in its 'neutral' state then

$$DW = 0.58, DD = 0.13, QSD = 0.0425, YW = 1.8, DSD = 0.07, YSD = 0.155$$
$$lnYMU = -3.283414, YW = 1.8, QMU = 0.047$$
hence $a \approx -YW \times QMU - lnYMU = 3.257$ and $k \approx 0.96$

k is around 0.96 even when model is some way away from its neutral position and is unlikely to fall below 0.9 and will never rise above 1. Hence we can approximate the random components in the log price movement as:

$$a_{1,QZ} \approx (DW \times DD + 1 - DW - 0.96 \times YW) \times QSD = -0.052$$
$$a_{1,DZ} = DSD = 0.070$$
$$a_{1,YZ} \approx -0.96 \times YSD = -0.149$$

<u>Fixed Interest Gilts (i=2)</u>

The only contribution to the $a_{2,Z}$ will be from the following term, which can be expanded approximately as:

$$\log\left(1 + \frac{1}{CM(t) + CMU \times \exp\left(CN(t)\right)}\right) = \log\left(1 + \frac{1}{a+x}\right) \approx \log\left(1 + \frac{1}{a}\right) - \frac{x}{a(a+1)}$$
$$where\, a + x \approx CD \times \left(QMU + QA \times \left(I(t-1) - QMU\right) + QSD \times QZ(t)\right)$$
$$+ (1-CD) \times CM(t-1)$$
$$+ CMU \times \left(1 + CA1 \times CN(t-1) + CY \times YSD \times YZ(t) + CSD \times CZ(t)\right)$$
$$\Rightarrow a \approx (QMU + CMU)$$
$$and\, x \approx CD \times QSD \times QZ(t) + CMU \times CY \times YSD \times YZ(t) + CMU \times CSD \times CZ(t)$$

If the model is in its 'neutral' state then

$$CD = 0.045, QSD = 0.0425, QMU = 0.047, CSD = 0.185, YSD = 0.155,$$
$$CY = 0.34, CMU = 0.0305$$

Hence we can approximate the random components in the log price movement as:

$$a_{2,QZ} = -\frac{CD \times QSD}{(QMU + CMU)(1 + QMU + CMU)} = -0.023$$
$$a_{2,YZ} = -\frac{CMU \times CY \times YSD}{(QMU + CMU)(1 + QMU + CMU)} = -0.019$$
$$a_{2,CZ} = -\frac{CMU \times CSD}{(QMU + CMU)(1 + QMU + CMU)} = -0.068$$

The calculations are less accurate for this asset category than for the others.

Cash (i=3)

The formula for log($BR(t)/BR(t$-1)) has no dependency on random components other than via state variables at time t-1. Hence all $a_{3,Z}$ are zero.

Index-linked Gilts (i=4)

We can expand the formula for log($RR(t)/RR(t$-1)) as follows:

$$\begin{aligned}\log\big(RR(t) / RR(t-1)\big) &= I(t) + \log\big(1 + \exp(-lnR(t)\big) + lnR(t-1) \\ &\approx lnR(t-1) + QMU + QA \times \big(I(t-1) - QMU\big) + QSD \times QZ(t) \\ &\quad + \log(1 + e^a) - \frac{e^a}{1+e^a}\big(RBC \times CSD \times CZ(t) + RSD \times RZ(t)\big)\end{aligned}$$
where $a = -lnRMU + RA \times (lnR(t-1) - lnRMU)$

If the model is in its 'neutral' state then

$$RBC = 0.22, DD = 0.13, QSD = 0.0425, YW = 1.8, CSD = 0.185, RSD = 0.05$$
$$lnRMU = -3.218875 \qquad \text{hence } a \approx 3.218875 \text{ and } k \approx 0.96$$

Hence we can approximate the random components in the log price movement as:

$$a_{4,QZ} = QSD = 0.0425$$
$$a_{4,CZ} = -0.96 \times RBC \times CSD = -0.96 \times 0.22 \times 0.185 = -0.039$$
$$a_{4,RZ} = -0.96 \times RSD = -0.96 \times 0.05 = -0.048$$

Property (i=5)

The log price movement can be expanded approximately as:

$$\begin{aligned}\log\big(AR(t)/AR(t-1)\big) &\approx c - k\times ZSD\times ZZ(t)\\ &+ EW\times ED\times QSD\times QZ(t) + (1-EW)\times QSD\times QZ(t)\\ &+ EBZ\times ZSD\times ZZ(t) + ESD\times EZ(t)\end{aligned}$$

where $k = \dfrac{e^a}{1+e^a}$ and $a \approx -lnZMU$

If the model is in its 'neutral' state then

$$\begin{aligned}&EW = 1, ED = 0.11, QSD = 0.0425, ESD = 0.06, ZSD = 0.12\\ &lnZMU = -2.6036, EBZ = 0.24 \qquad \text{hence } a \approx 2.6036 \text{ and } k \approx 0.93\end{aligned}$$

Hence we can approximate the random components in the log price movement as:

$$\begin{aligned}a_{5,QZ} &= (EW\times ED + 1 - EW)\times QSD = 0.0047\\ a_{5,EZ} &= ESD = 0.060\\ a_{5,ZZ} &= (-0.93 + EBZ)\times ZSD = -0.083\end{aligned}$$

C.8 The obvious way to make the Wilkie model a continuous time model is to assume that over the period t-1 to t the random components $YZ(t)$, ... behave like unit Brownian motions, with standard deviation over time h of $\sqrt{h}$. The quadratic variation accruing over an infinitesmal period h for a total return option linking asset category i with asset category j is then:

$$C(t+h) - C(t) = \big((Z_i - E(Z_i)) - (Z_j - E(Z_j))\big)^2$$

$$where\ Z_i = \log\left(\frac{R_i(t+h)}{R_i(t)}\right)\ and\ Z_j = \log\left(\frac{R_j(t+h)}{R_j(t)}\right)$$

C.9 Since the normal error terms $YZ(t)$, ... are independent of each other we have:

$$C(t+h) - C(t) \approx \big((a_{i,QZ} - a_{QZ})^2 + (a_{i,DZ} - a_{j,DZ})^2 + \ldots\big)h = \sigma_{i,j}^2 h, \text{say}$$

Hence C(T)-C(t) is approximately $\sigma_{i,j}(T-t)$. But we can express $\sigma_{i,j}^2$ as $s_i c_{ij} s_j$ for some suitable s_i and c_{ij}, i.e. as if it were a covariance matrix linking asset i to asset j. This means that the risk neutral probability distribution is approximately multi-variate log-normal, with this same covariance matrix, which is:

sector	standard deviation relative to cash* (% p.a.)	correlations				
		cash	fixed int.	ILG's	property	equities
cash	0.0	-				
fixed int.	7.4	-	1.00			
ILG's	7.5	-	0.30	1.00		
property	10.3	-	-0.01	0.03	1.00	
equities	17.3	-	0.31	-0.17	-0.01	1.00

* of log price movements *relative to the cash log price movement.*

C.10 The value of any total return put option in the Wilkie model framework (made continuous as described above) is then approximately given by the following formula (where $\sigma_{i,j}$ is as above):

$$P(S,t) = E.N(-d_2) - S.N(-d_1)$$

$$\text{where } d_1 = \frac{\log(S/E) + \sigma_{i,j}^2 (T-t)/2}{\sigma_{i,j}\sqrt{T-t}} \qquad d_2 = d_1 - \sigma_{i,j}\sqrt{T-t}$$

C.11 The $\sigma_{i,j}$ seem to differ a little from those simulated in Wilkie (1995) over a one year period starting in the Wilkie model's 'neutral' position. The reasons seem to be:

(a) the simulations in Wilkie (1995) are of price movements not log price movements and will inevitably contain sampling errors.

(b) the continuous time analogue described in Section C.8 may not perfectly compound over a year to the discrete yearly model. Any imperfections arising from this could easily be accommodated by an appropriate (modest) alteration to the $\sigma_{i,j}$ given above.